Northeastern University
Khoury College of
Computer Sciences

Recap

DS 4400 | Machine Learning and Data Mining I

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Spring 2026

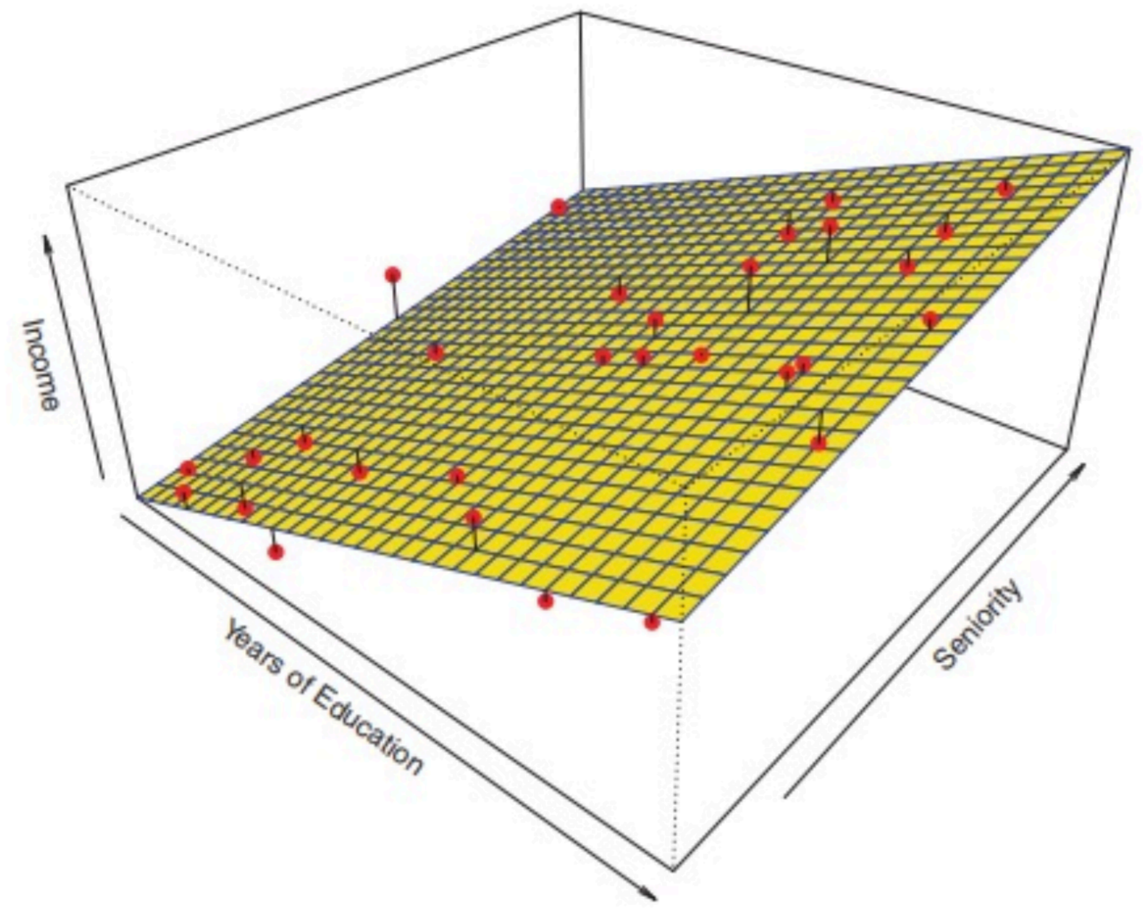
Monday | January 26, 2026

Linear Regression

Linear Regression

Linear Regression

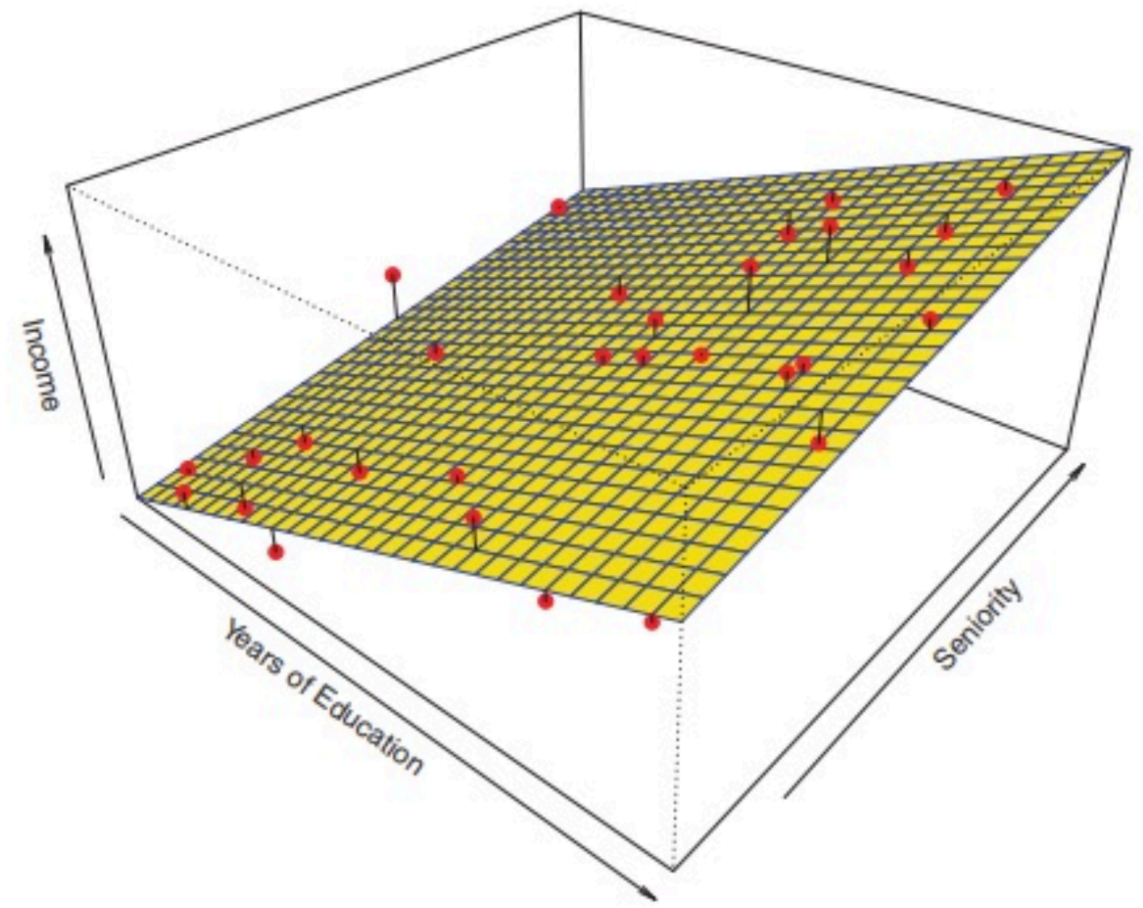
Linear Regression



- Linear Model

$$f_\theta(x) = \theta_0 + \theta_1 x_0 + \theta_2 x_1$$

Linear Regression



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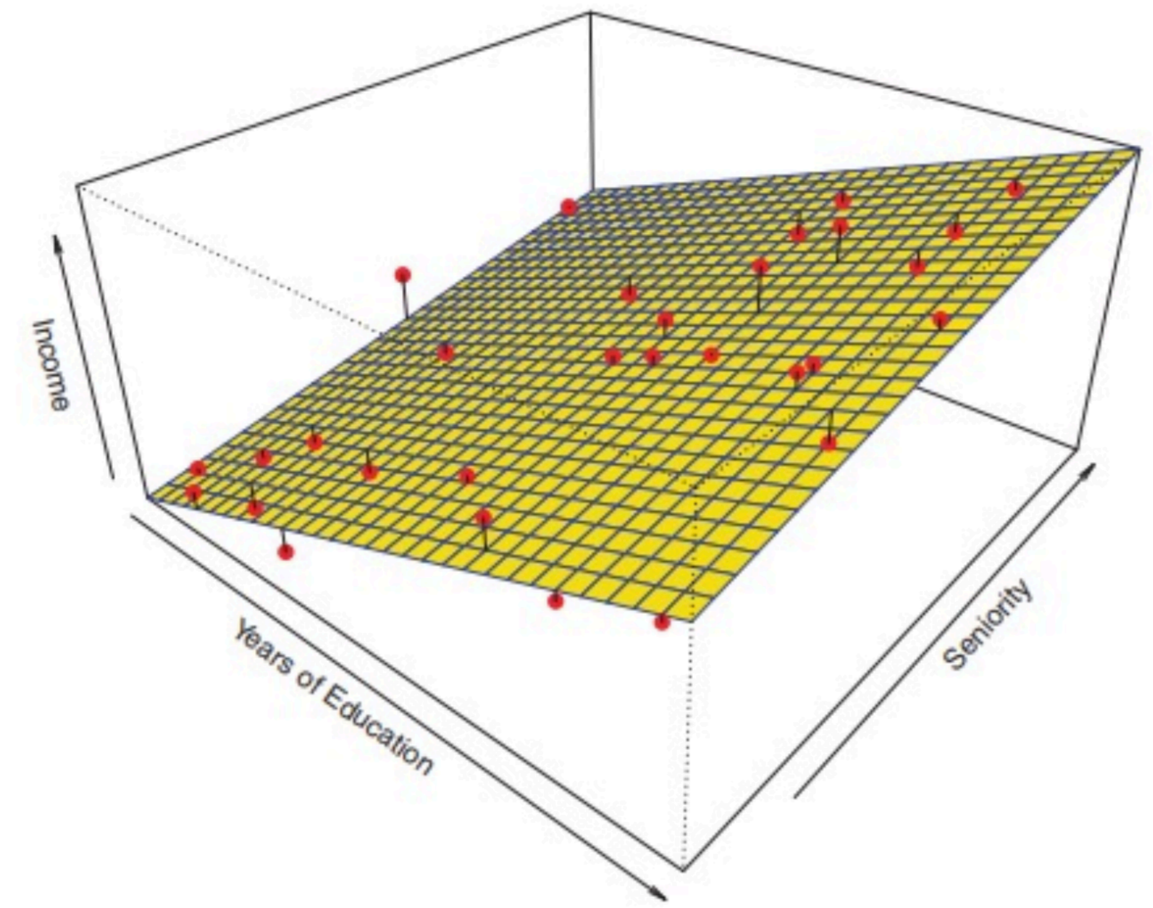
Learnable parameters

Linear Regression



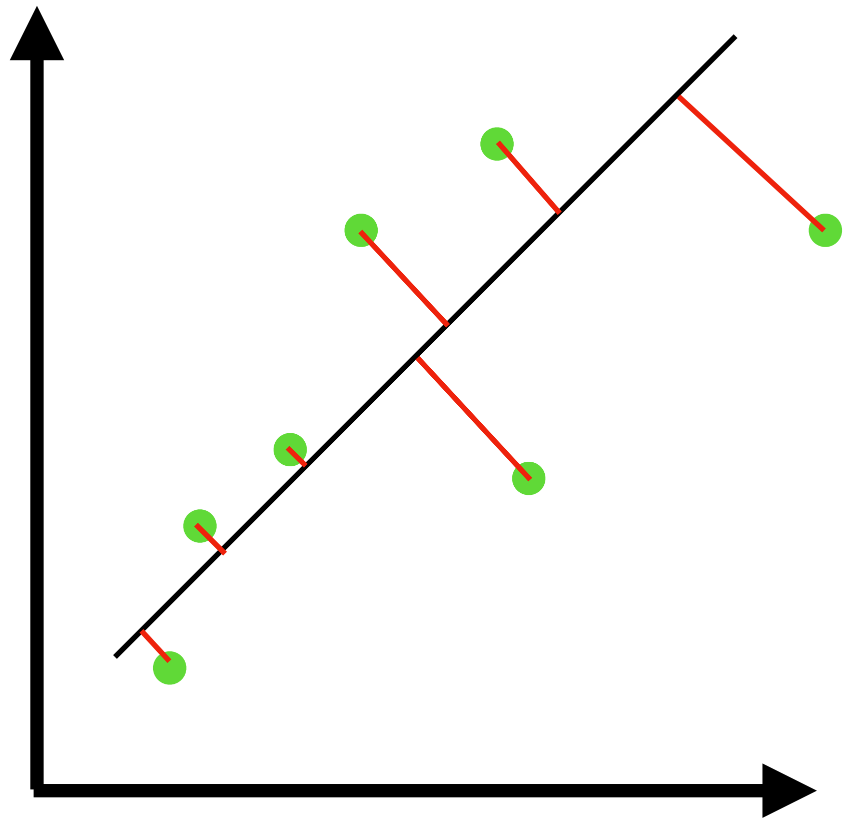
- Linear Model

$$f_\theta(x) = \theta_0 + \theta_1 x_0 + \theta_2 x_1$$

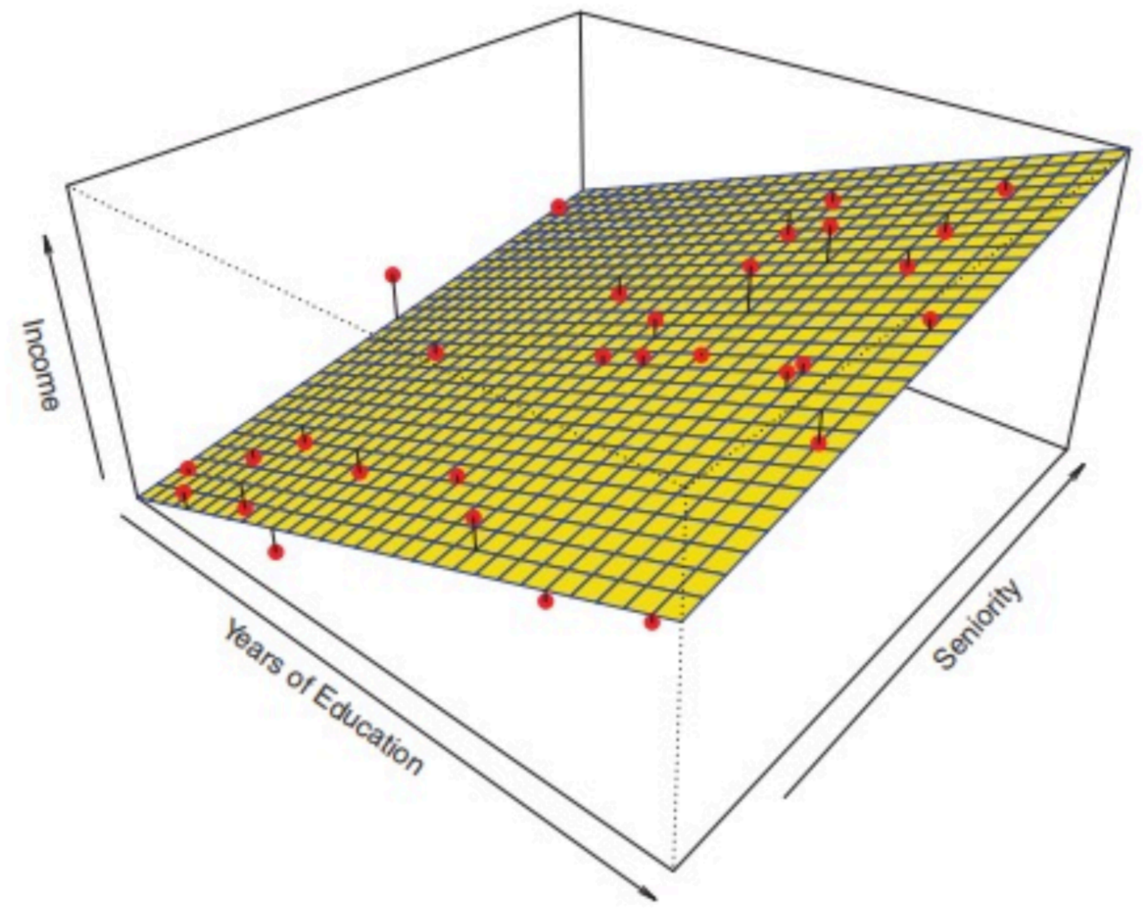
- Loss Functions (also called Cost Functions)

The red lines are called residuals

$$L(\theta) = \frac{1}{m} \sum_{i=1}^{m} [f_\theta(x_i) - y_i]^2$$ - Mean Squared Error



Linear Regression



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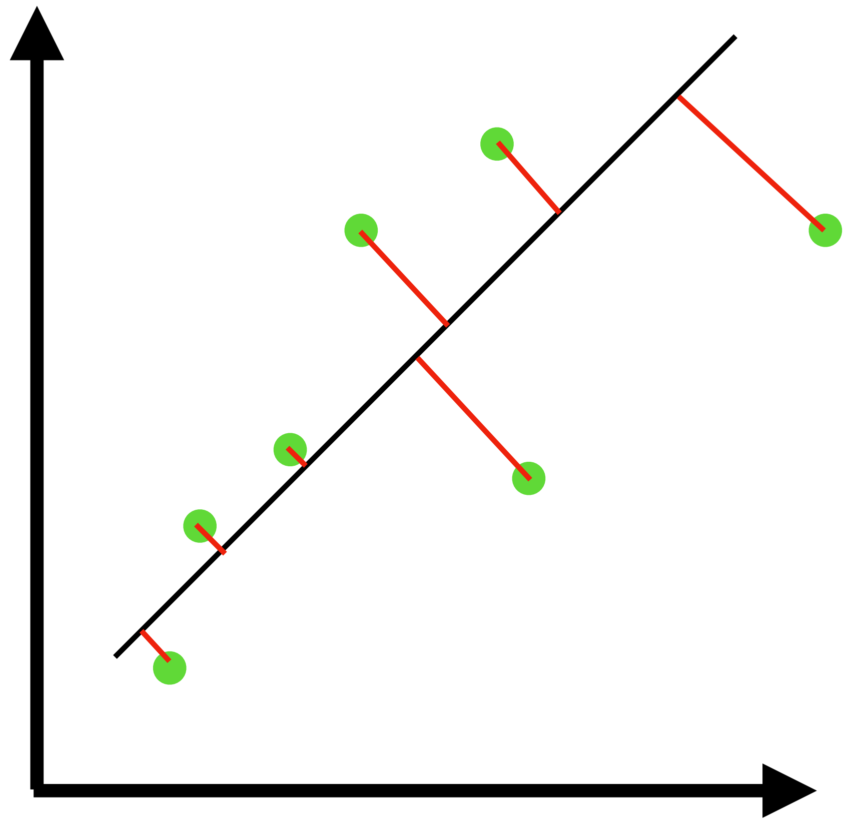
$$L(\theta) = \sum_{i=1}^{m}[f_\theta(x_i) - y_i]^2$$ - Residual Sum of Squares



Linear Regression

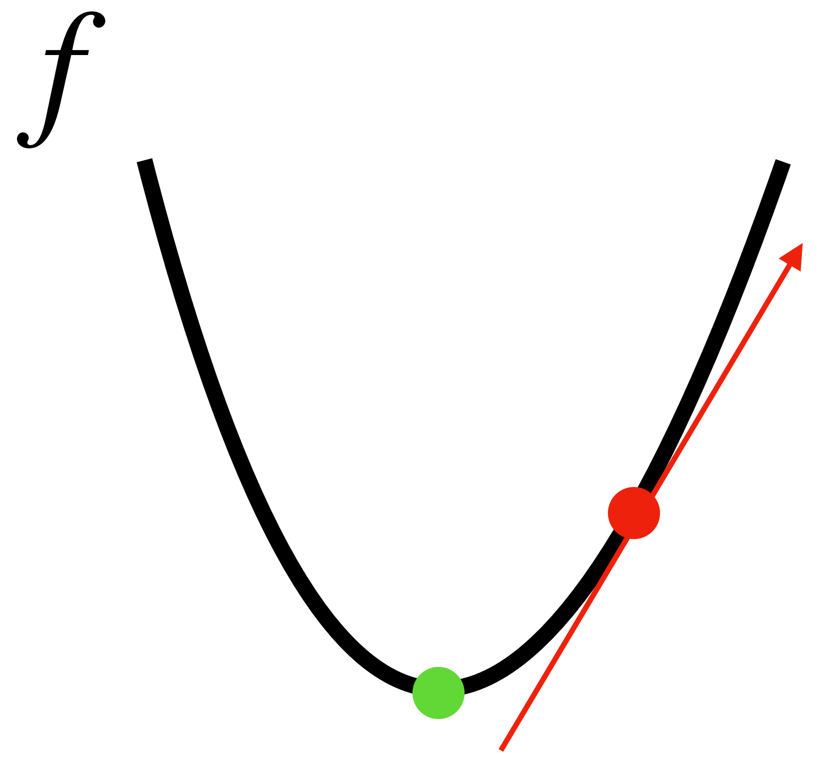
- Linear Model

$$f_\theta(x) = \theta_0 + \theta_1 x$$

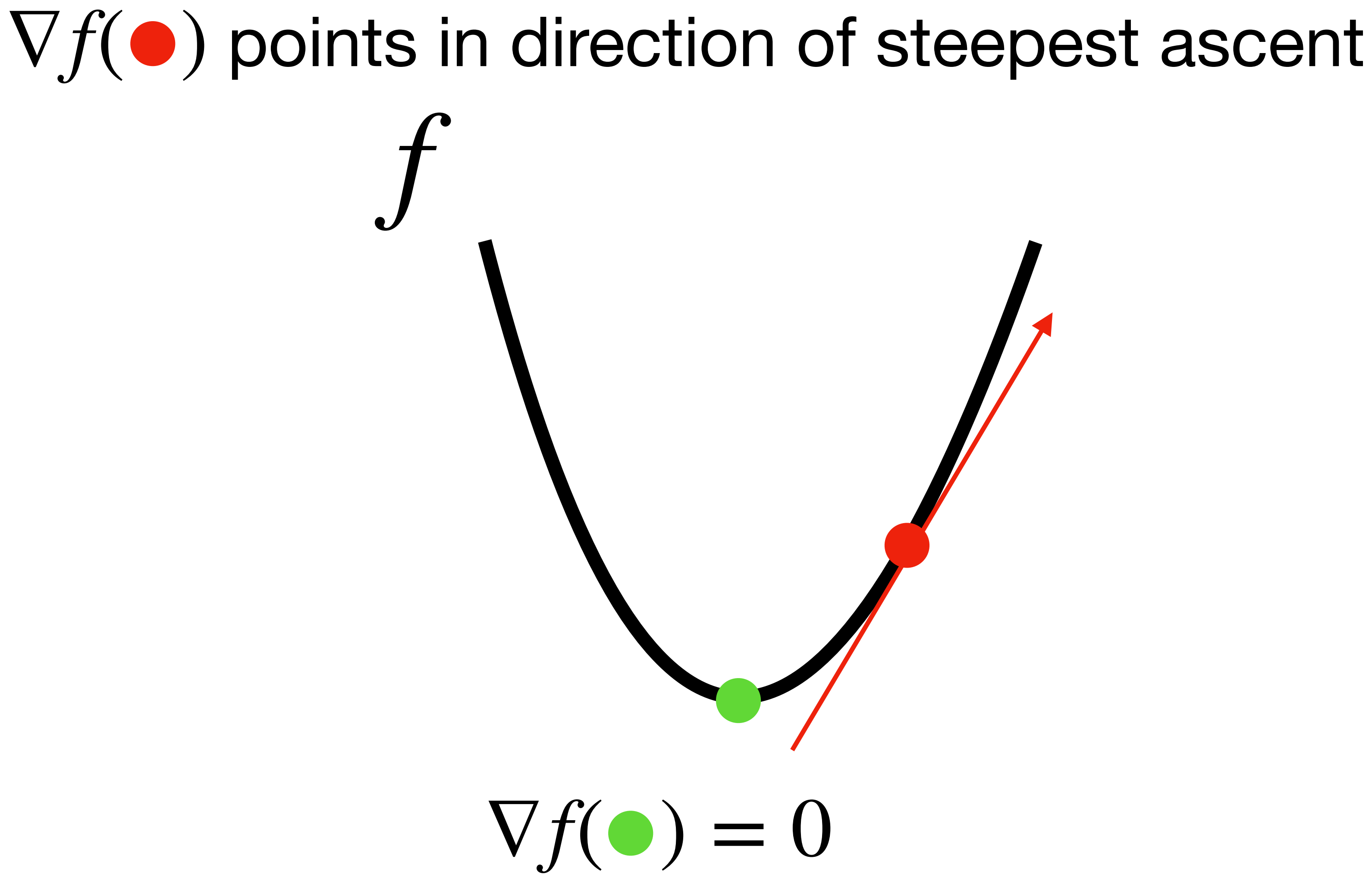
- How do we find the solution to this? How do we find the optimal θ?
  - We optimize θ to minimize the loss function

$$L(\theta) = \frac{1}{m}\sum_{i=1}^{m}[f_\theta(x_i) - y_i]^2$$

$$L(\theta) = \frac{1}{m}\sum_{i=1}^{m}[\theta_0 + \theta_1 \cdot x - y_i]^2$$



Linear Regression



Linear Regression

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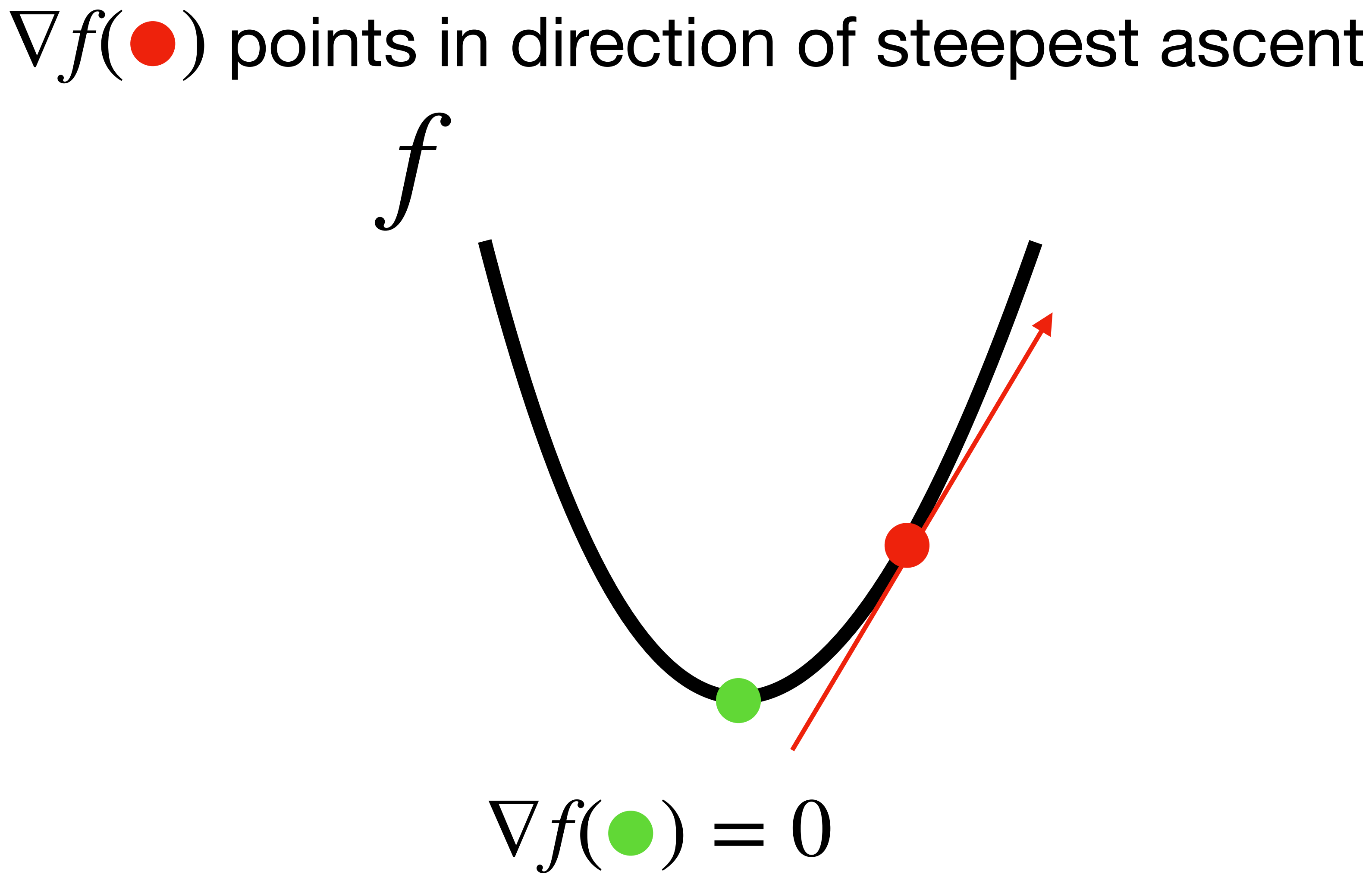
$$L(\theta) = \frac{1}{m}\sum_{i=1}^{m}[\theta_0 + \theta_1 \cdot x_i - y_i]^2$$

Find the point where $\nabla L(\theta) = 0$

$$\frac{\partial L(\theta)}{\partial \theta_0} = \frac{2}{m}\sum_{i=1}^{m}(\theta_0 + \theta_1 x_i - y_i) = 0$$

$$\frac{\partial L(\theta)}{\partial \theta_1} = \frac{2}{m}\sum_{i=1}^{m}x_i \cdot (\theta_0 + \theta_1 x_i - y_i) = 0$$



Linear Regression

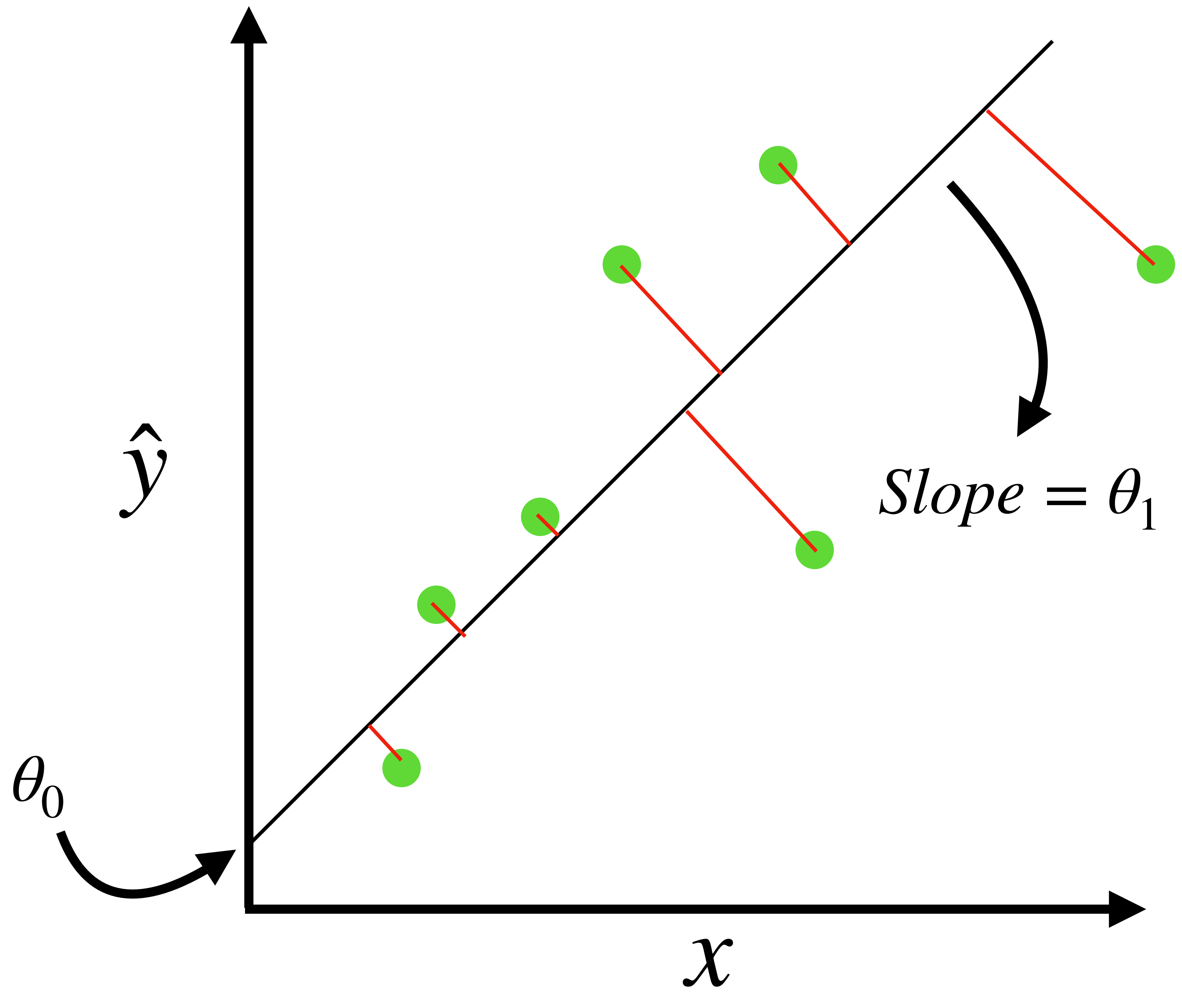
$$\theta_0 = \bar{y} - \theta_1 \bar{x}$$

$$\theta_1 = \frac{Cov(x, y)}{Var(x)}$$

The slope $\theta_1 = \dfrac{Cov(x, y)}{Var(x)}$ makes sense:

- If x and y covary strongly (move together), the slope is steeper
- If x has high variance (spread out), the slope is gentler
- The **sign** of covariance determines if the line goes up or down

$$f_\theta(x) = \theta_0 + \theta_1 x$$



Linear Regression

Solutions in Matrix Form

Linear Regression

Solutions in Matrix Form

- Let's look at the matrix formulation of the same problem

$$L(\theta) = \frac{1}{m}\sum_i (y_i - \hat{y}_i)^2$$

But in matrix form, $f_\theta(x) = \hat{Y} = X\theta$, where $X \in \mathbb{R}^{m \times d}$ has m rows of data and d columns of features and $\theta = \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} \in \mathbb{R}^{d \times 1}$

$$L(\theta) = \frac{1}{m}\sum (Y - X\theta)^2$$

(think back to system of equations for why this is true)

Quick Recap

Systems of Linear Equations - Linear Regression Example

- Consider the equation $y = w_0 x_0 + w_1 x_1$

y Price	x_0 # Rooms	x_1 Sq. Ft.
2000	1	450
2100	1	510
2400	2	980
3000	3	1500

$$(1) \cdot w_0 + (450) \cdot w_1 = 2000$$

$$(1) \cdot w_0 + (510) \cdot w_1 = 2100$$

$$(2) \cdot w_0 + (980) \cdot w_1 = 2400$$

$$(3) \cdot w_0 + (1500) \cdot w_1 = 3000$$

$$\longrightarrow$$

$X \in \mathbb{R}^{4\times2}$ $\quad W \in \mathbb{R}^{2\times1}$ $\quad y \in \mathbb{R}^{4\times1}$

$$\begin{bmatrix} 1 & 450 \\ 1 & 510 \\ 2 & 980 \\ 3 & 1500 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 2000 \\ 2100 \\ 2400 \\ 3000 \end{bmatrix}$$

Linear Regression
Solution

We want to find the minimum so set gradient to zero

$$\nabla L(\theta) = -2X^TY + 2X^TX\theta = 0$$

$$2X^TX\theta = 2X^TY$$

$$X^TX\theta = X^TY$$

If X^TX is invertible, then

$$\theta = (X^TX)^{-1}X^TY$$

Practical Issues in Linear Regression

Multicollinearity

- When two features are highly correlated or are linearly dependent on each other

Practical Issues in Linear Regression

Multicollinearity

- When two features are highly correlated or are linearly dependent on each other
- Why it's a problem: $\theta = (X^T X)^{-1} X^T Y$
    - $X^T X$ becomes nearly singular (ill-conditioned)
    - Small changes in data cause huge changes in coefficients
    - Coefficients become unreliable and hard to interpret
    - Standard errors blow up

Practical Issues in Linear Regression

Multicollinearity

- When two features are highly correlated or are linearly dependent on each other
- Why it's a problem: $\theta = (X^T X)^{-1} X^T Y$

Simple Detection:
If correlation between features ≥ 0.8

  - $X^T X$ becomes nearly singular (ill-conditioned)
  - Small changes in data cause huge changes in coefficients
  - Coefficients become unreliable and hard to interpret
  - Standard errors blow up

Practical Issues in Linear Regression

Quick Aside

$$\theta = \boxed{(X^T X)^{-1}} X^T Y$$

When else is this not going to be invertible?

$$X \in \mathbb{R}^{m \times n}$$

m: Number of training examples

n: Number of parameters in the model

Practical Issues in Linear Regression

Quick Aside

$$X \in \mathbb{R}^{m \times n}$$

$$\theta = (X^T X)^{-1} X^T Y$$

m: Number of training examples
n: Number of parameters in the model
$rank(X) = min(m, n)$

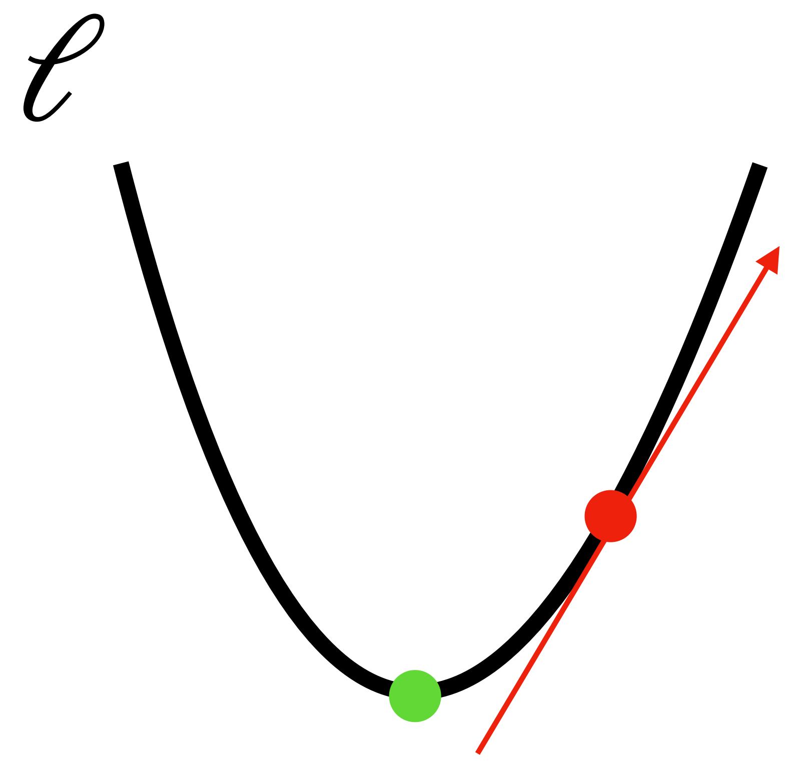
When else is this not going to be invertible?

If $m < n$, then $rank(X) \leq m$, so need **more** data points than number of parameters to get a unique set of parameters

Gradient Descent: Optimizing Loss Functions

- For any loss function $\ell(\theta)$
  - To find minimum, set $\nabla \ell = 0$ and solve for θ

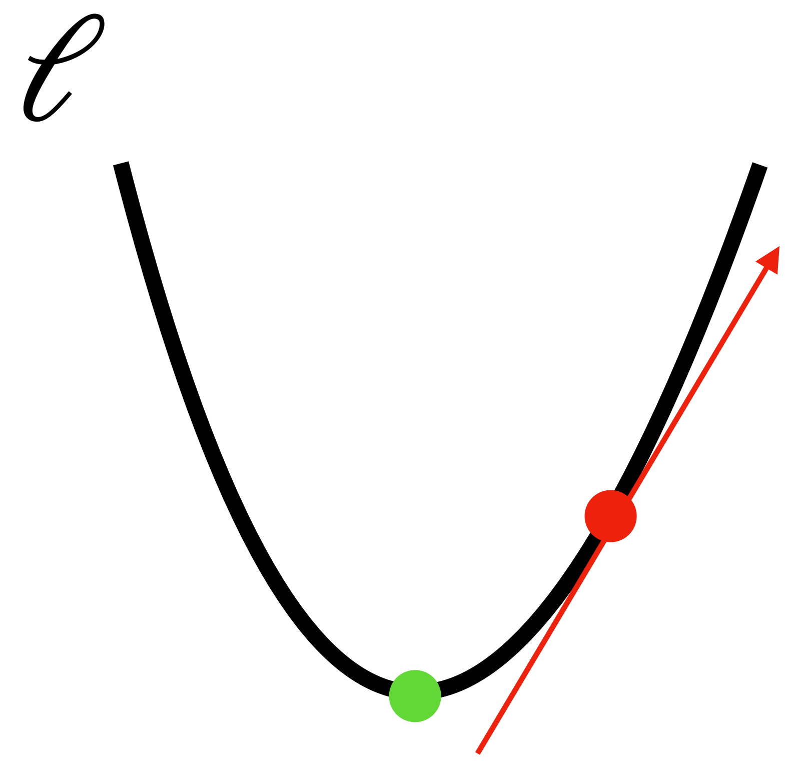
$\nabla \ell(\bullet)$ points in direction of steepest ascent



$\nabla \ell(\bullet) = 0$

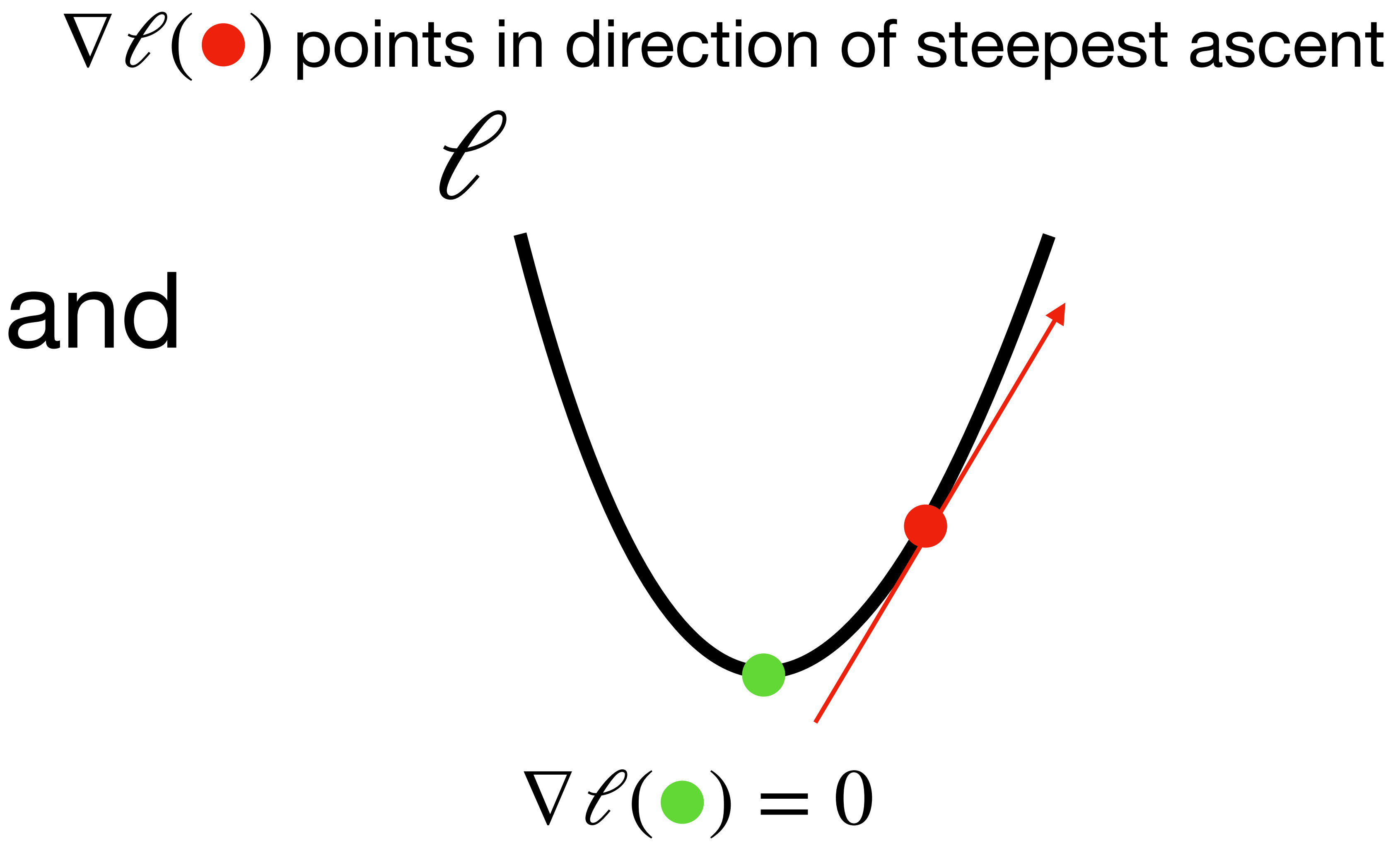
Optimizing Loss Functions

- For any loss function $\ell(\theta)$
  - To find minimum, set $\nabla \ell = 0$ and solve for θ
  - This is called the **closed form solution**
  - But it's not always possible to find closed form solutions, especially when there are a large number of parameters
  - Inverting a matrix is a costly operation - most common methods have complexity $O(n^3)$



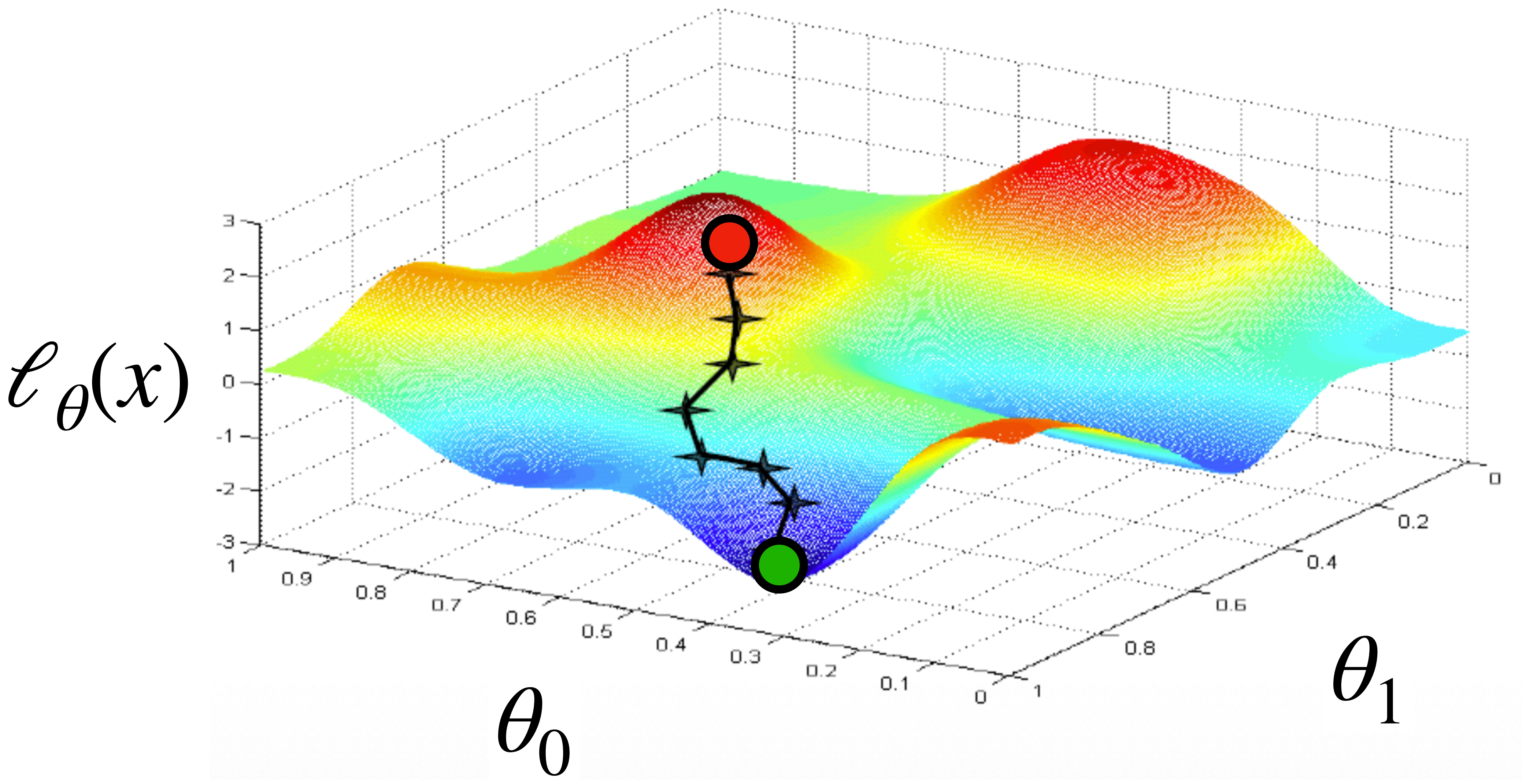
Optimizing Loss Functions

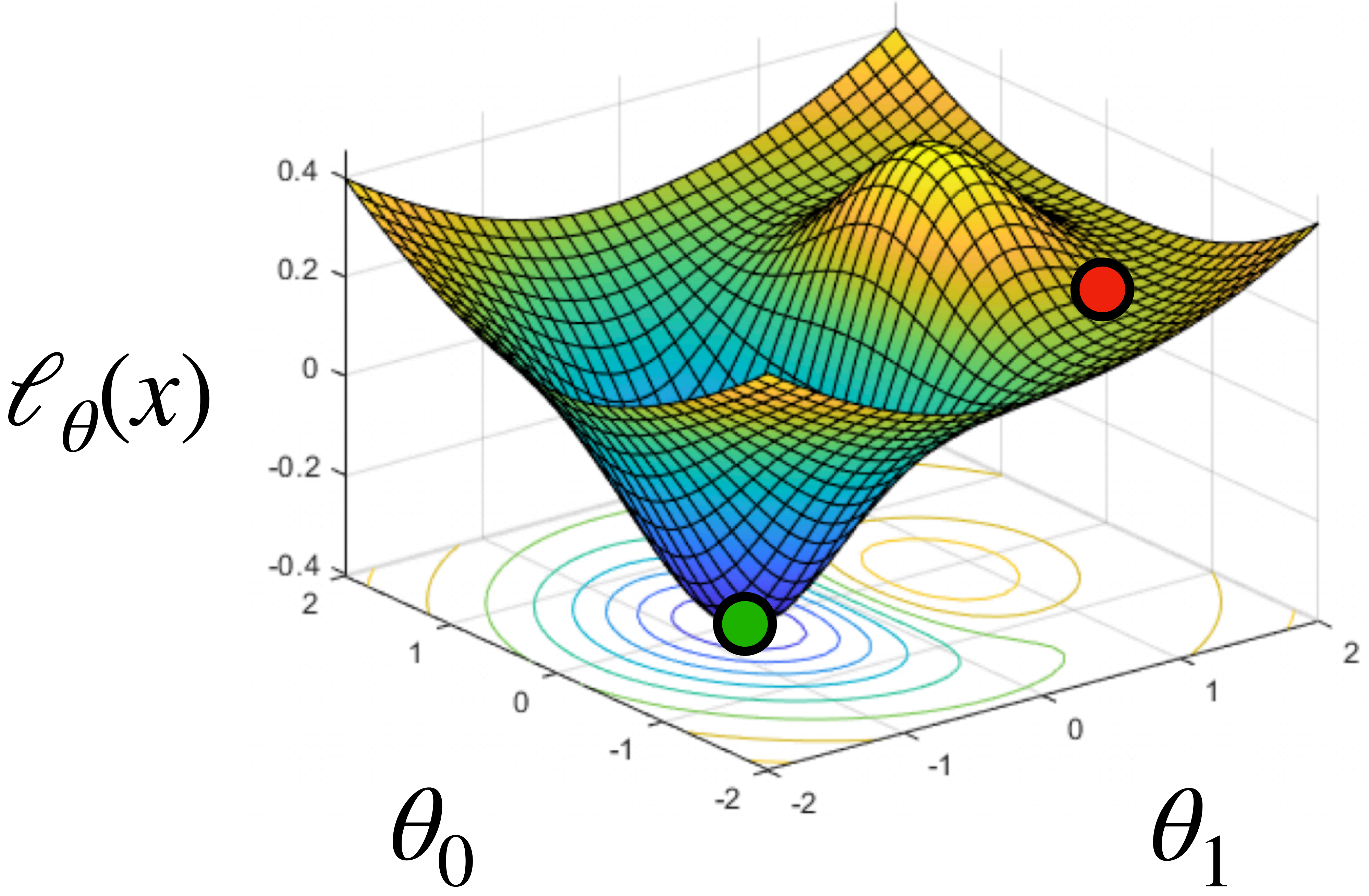
- This is where Gradient Descent comes in
  - Practical and efficient - has $O(mTn)$ where m is number of training points, T is number of epochs and n is number of features
  - Generally applicable to different loss functions
  - Convergence guarantees for certain types of loss functions (e.g., convex functions)



Optimizing Loss Functions

- What does the loss landscape look like with multiple learnable parameters?



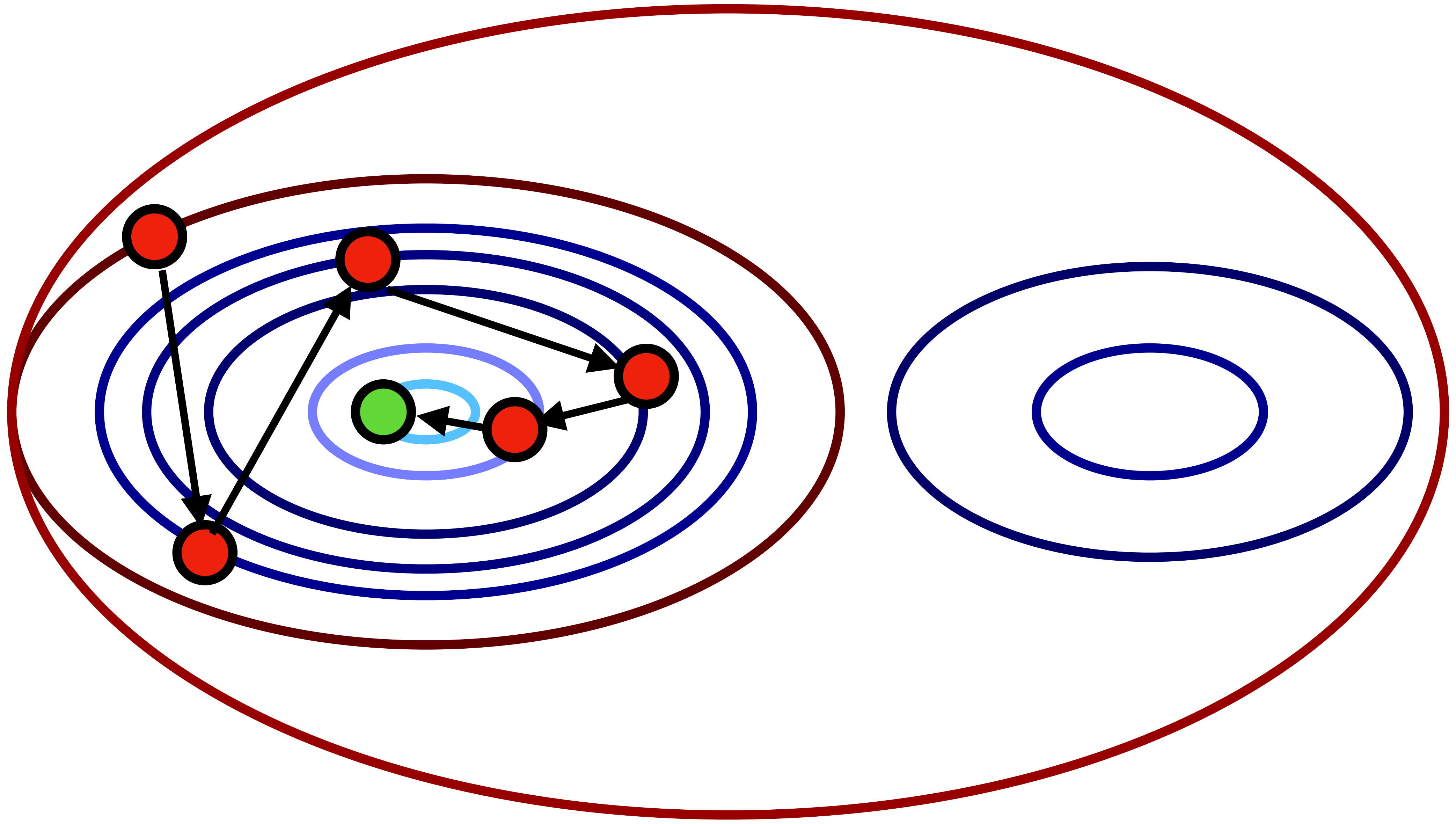


Optimizing Loss Functions

Optimizing Loss Functions

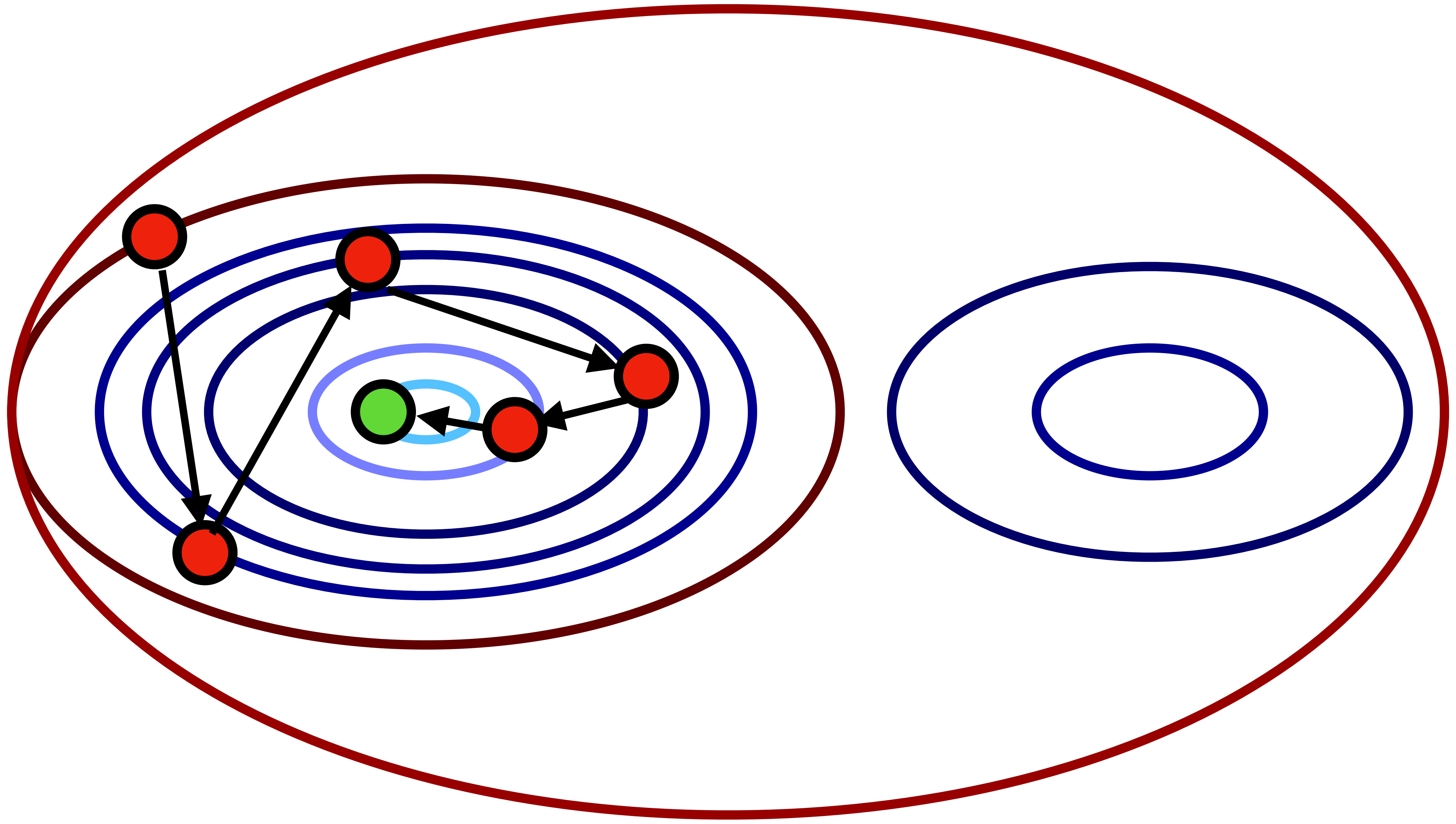
Gradient Descent - Formulation

$$\ell_\theta(x) = \frac{1}{m}\sum_i (y_i - \theta_0 - \theta_1 x_i)^2$$



Optimizing Loss Functions

Gradient Descent - Formulation



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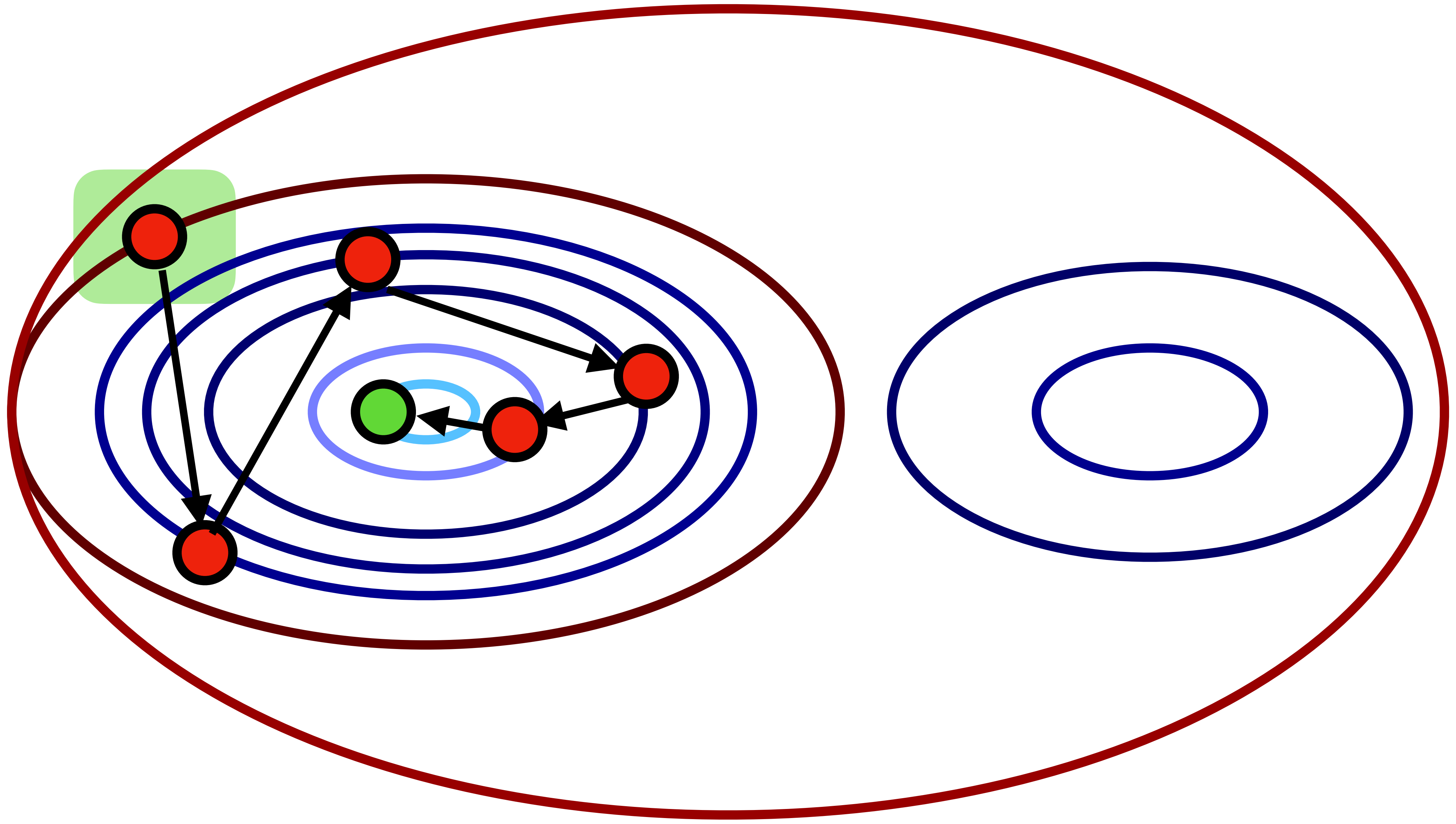
Step 1: Initialize θ_0, θ_1

Optimizing Loss Functions

Gradient Descent - Formulation



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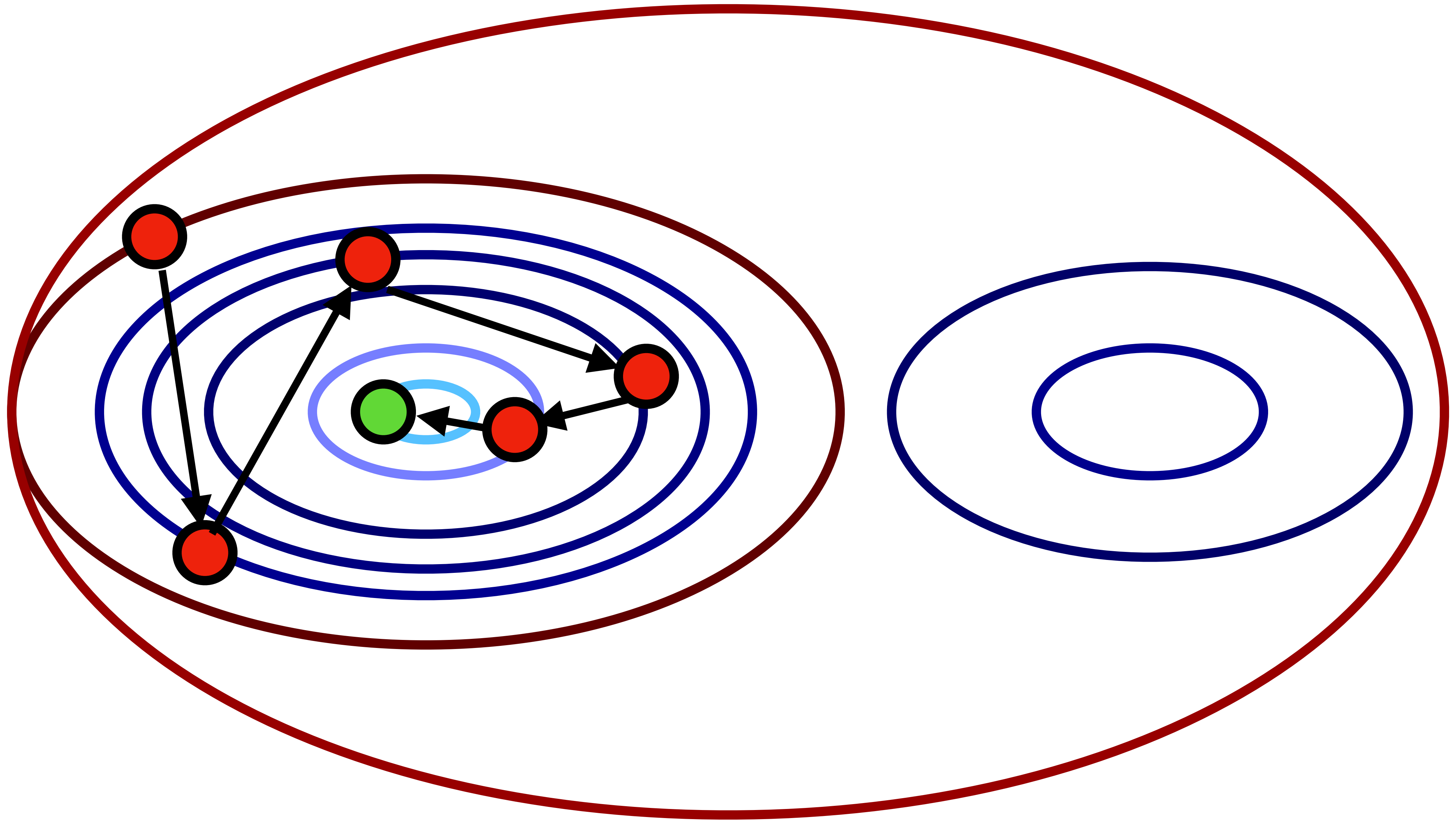
This is going to be your "starting point" on the loss landscape

Optimizing Loss Functions

Gradient Descent - Formulation



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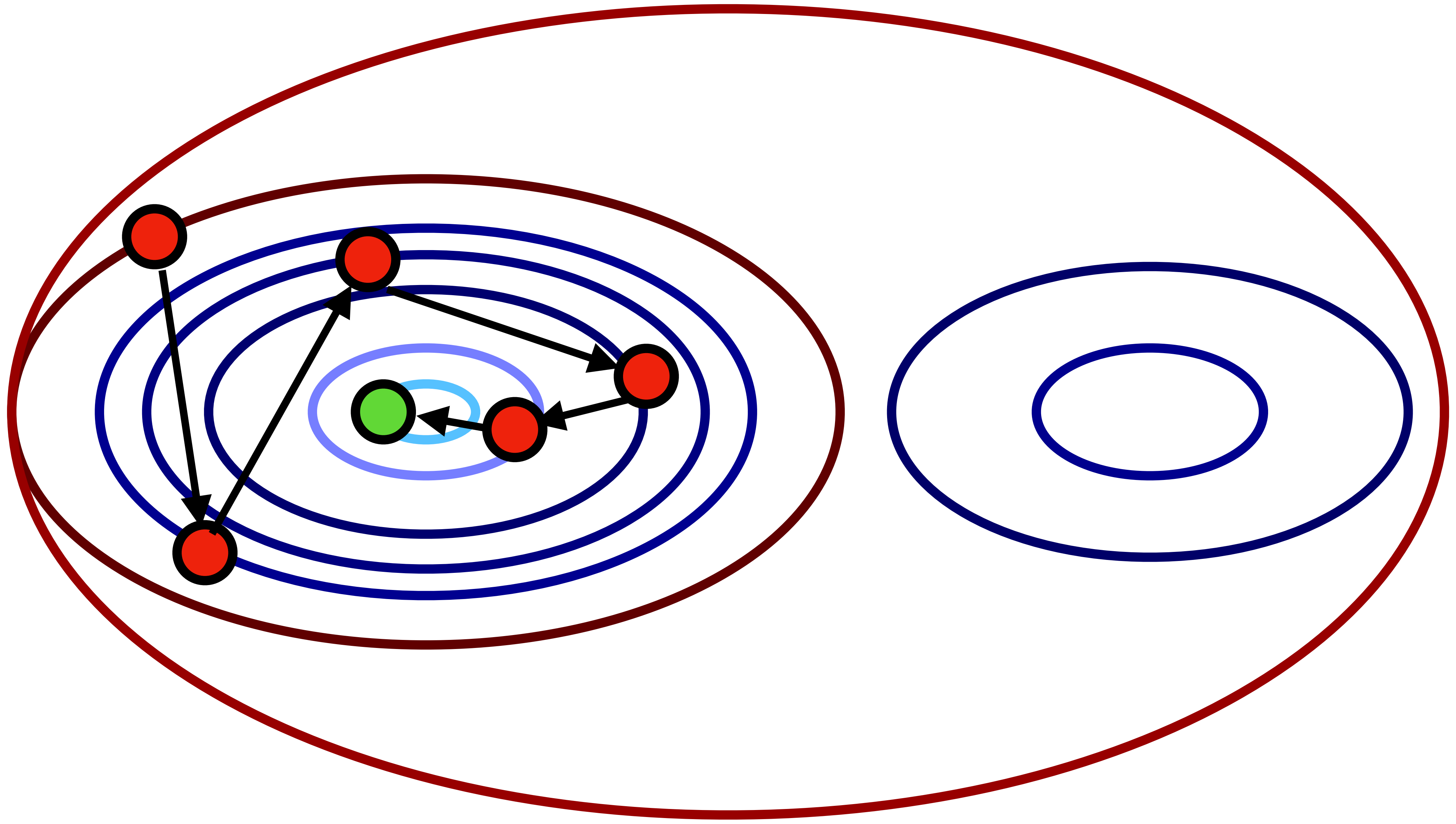
Step 2: Repeat Until Convergence

$$\theta_j \leftarrow \theta_j - \alpha \cdot \frac{\partial \ell_\theta(x)}{\partial \theta_j}$$

Optimizing Loss Functions

Gradient Descent - Formulation



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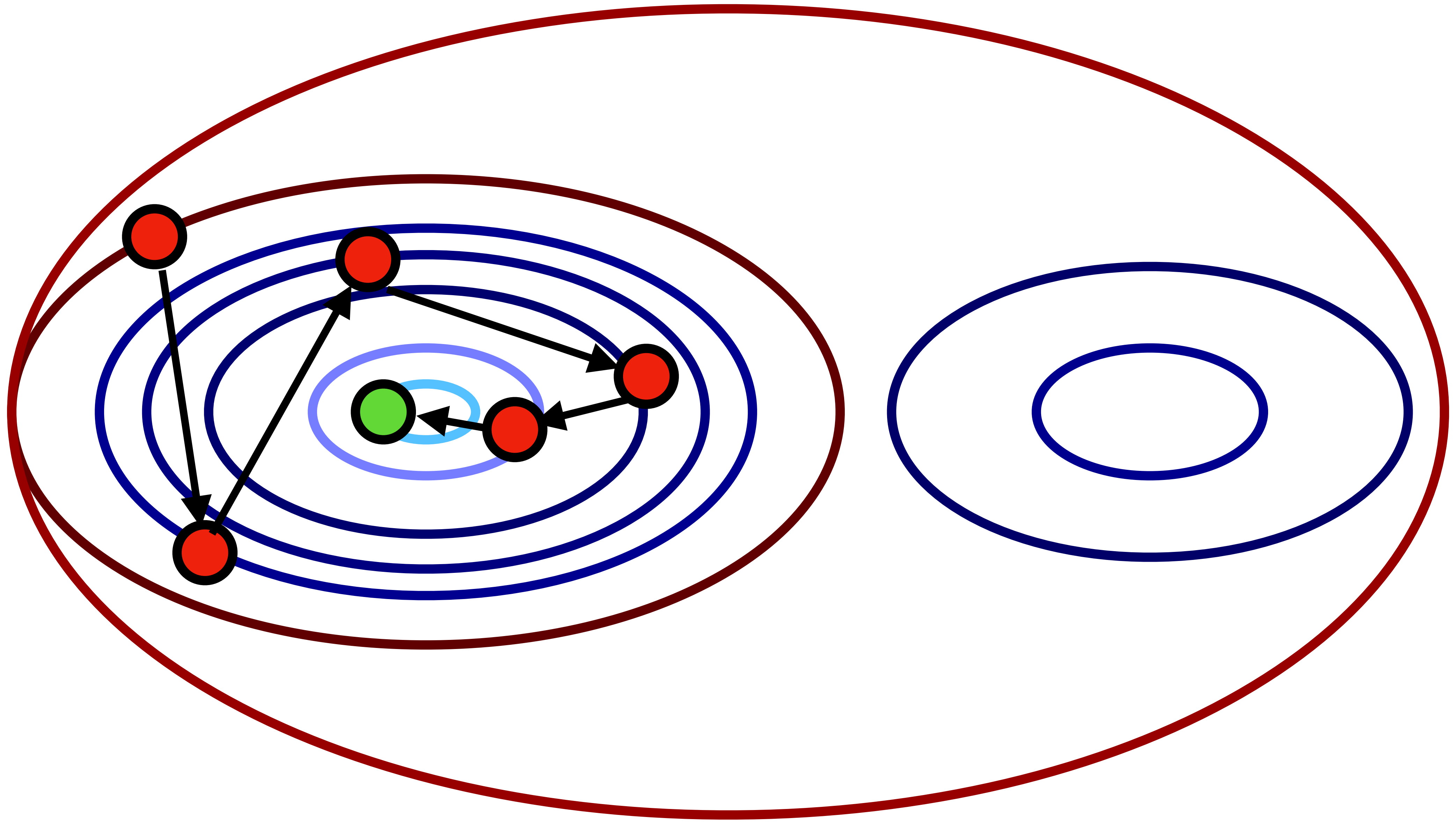
Negative of partial derivative points in the direction of steepest descent

Optimizing Loss Functions

Gradient Descent - Formulation



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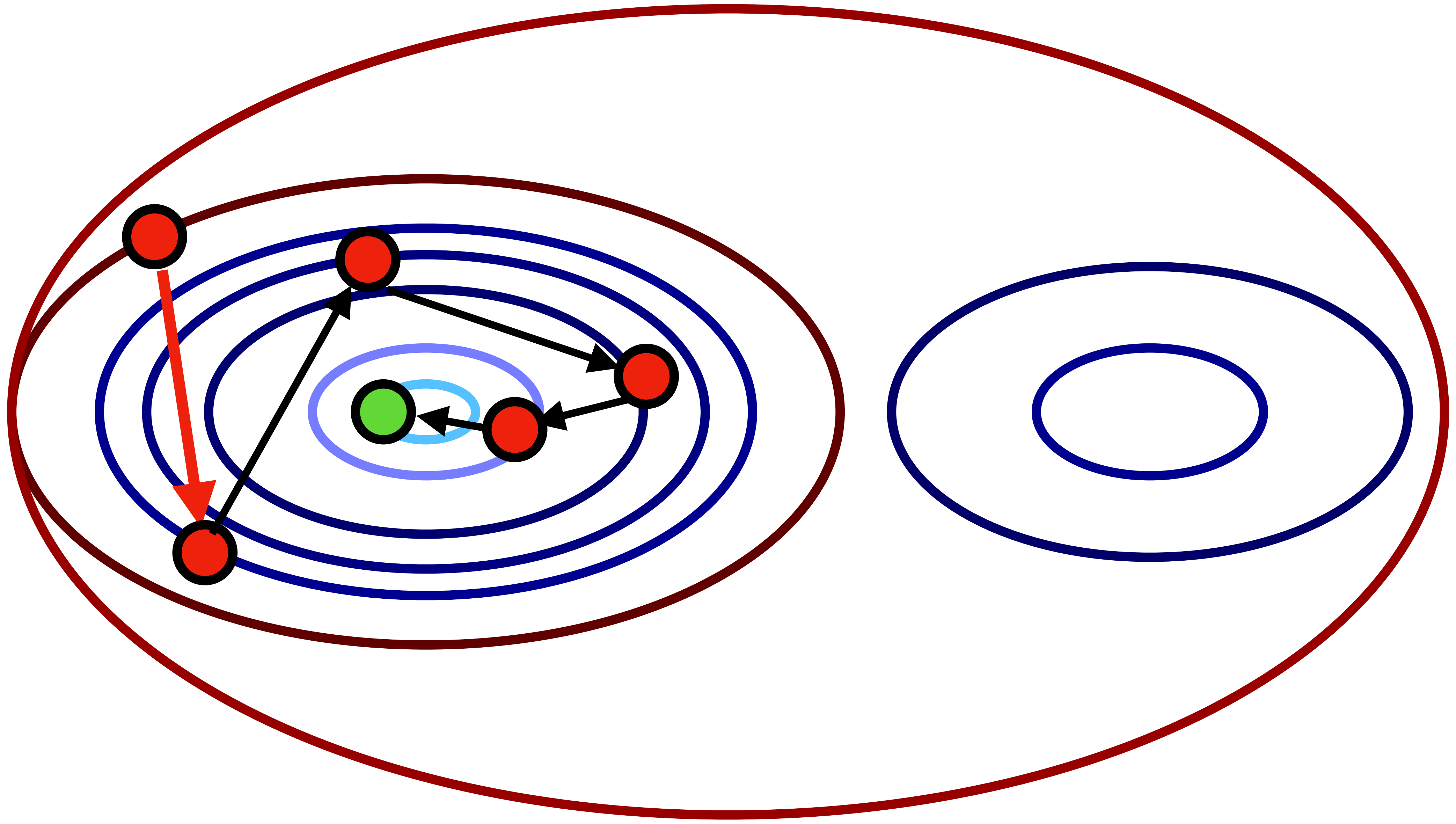
α : Learning Rate

Optimizing Loss Functions

Gradient Descent - Formulation

α **controls how big a step to take**



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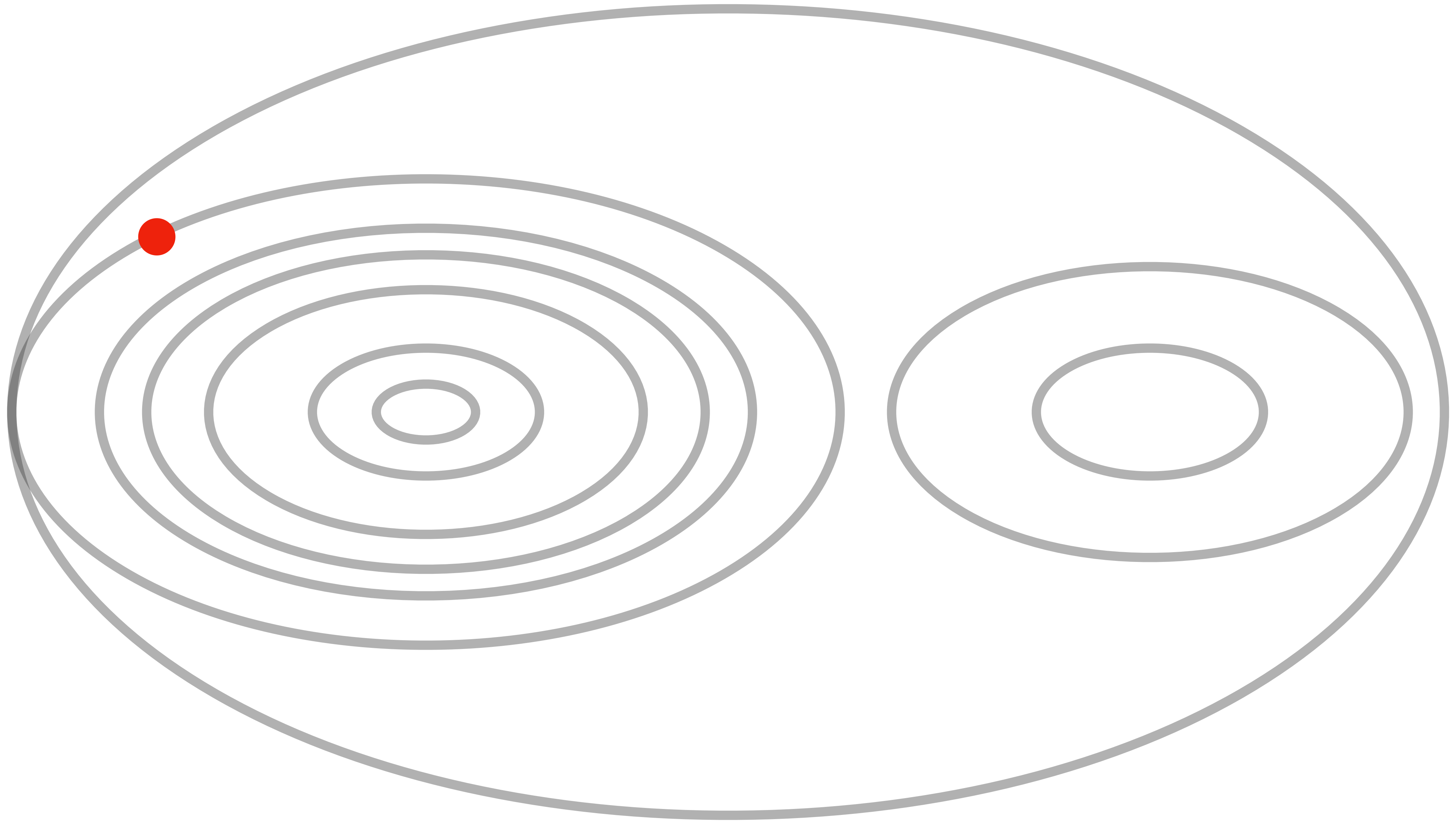
Optimizing Loss Functions

Gradient Descent - Effect of Learning Rate

What happens when α is too small?
Say $\alpha = 10^{-5}$



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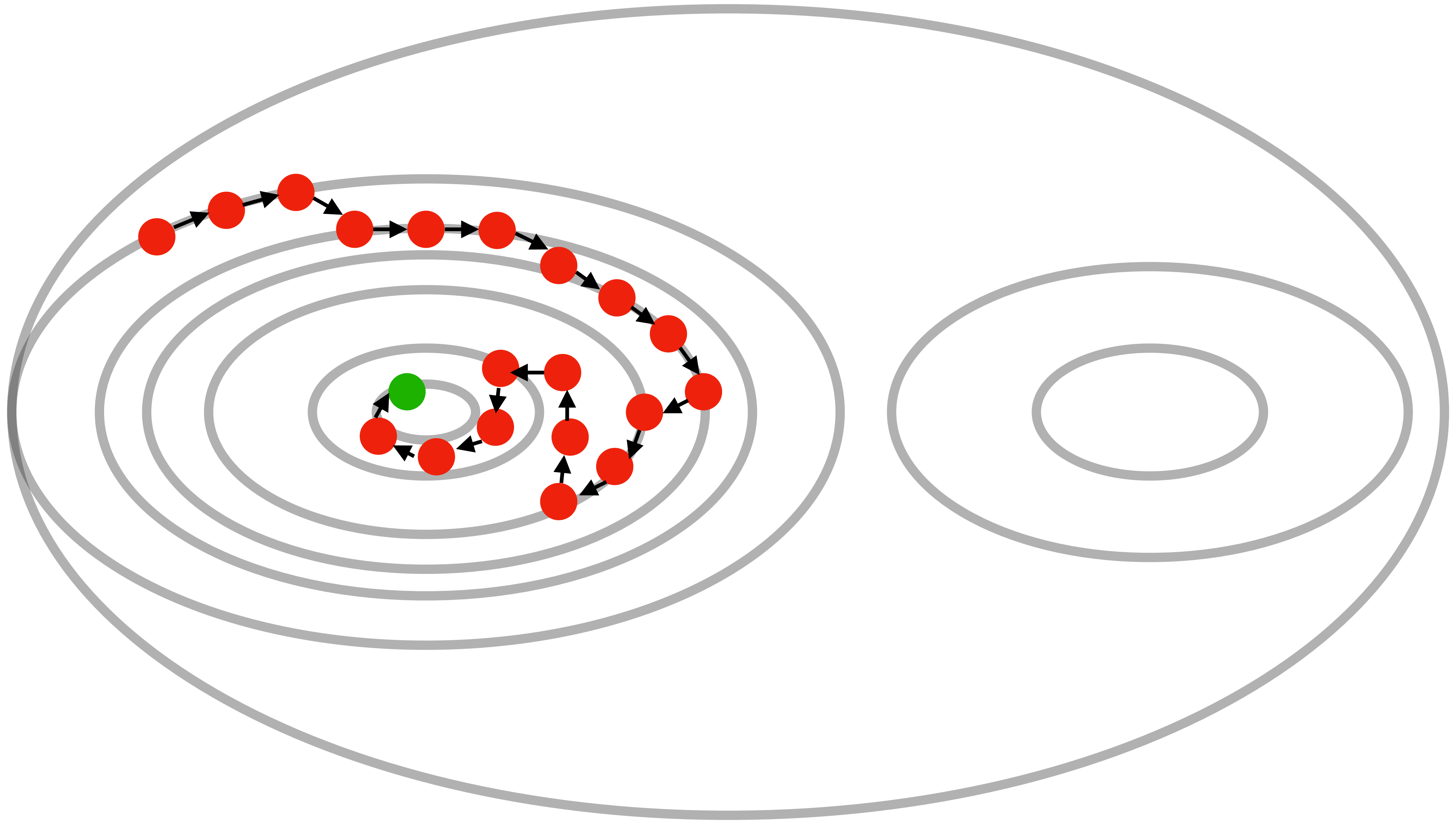

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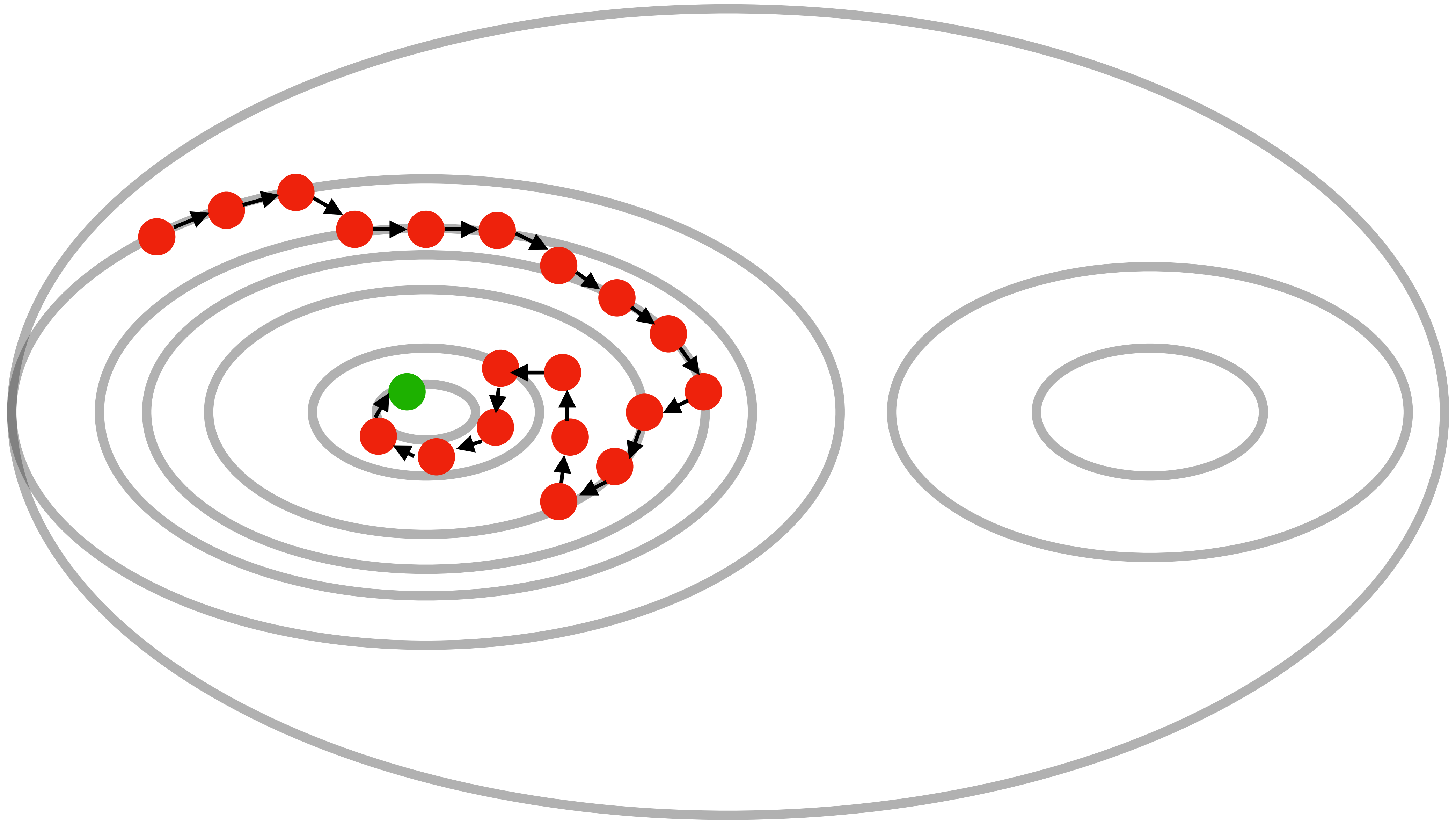

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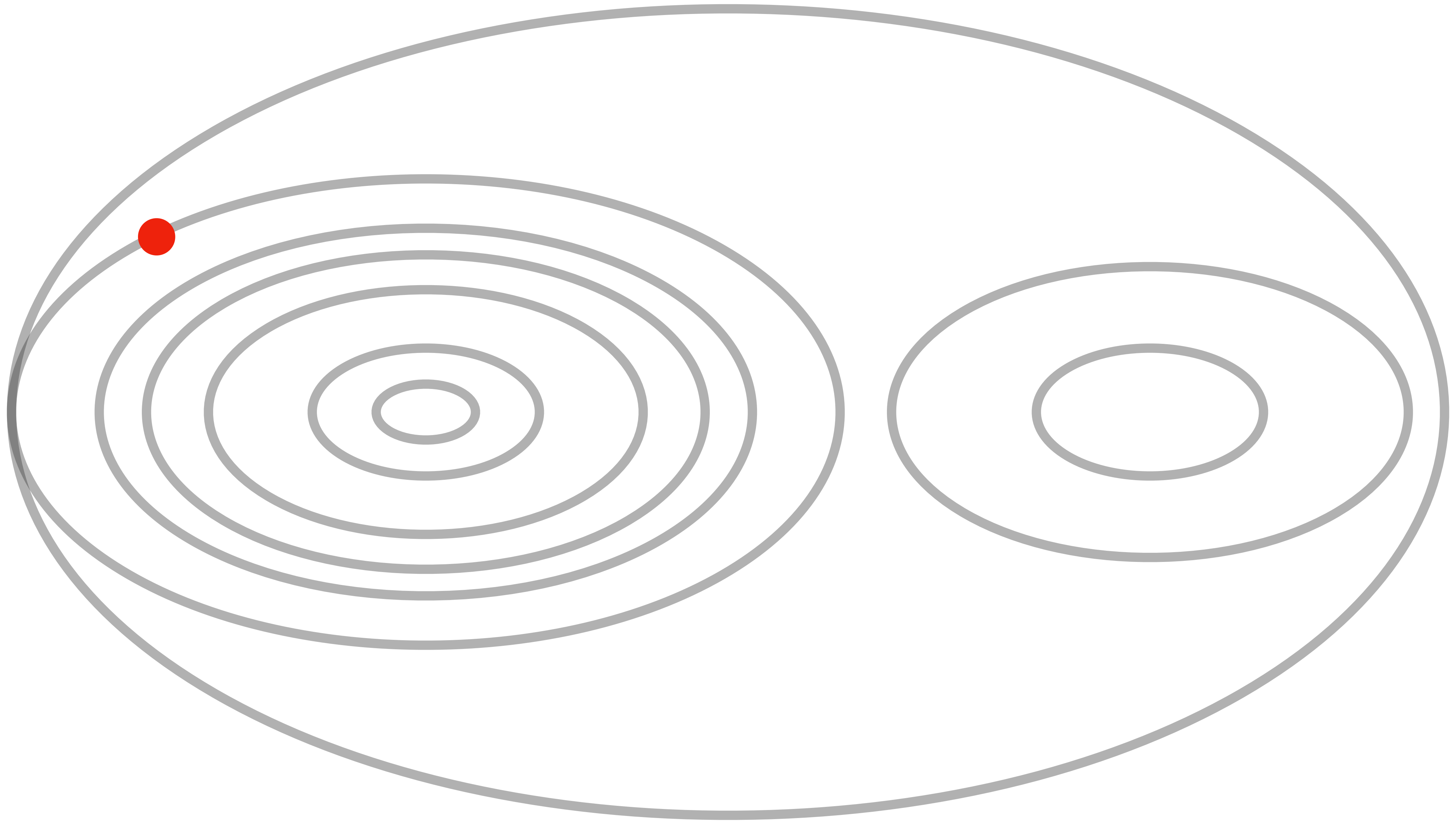

Optimizing Loss Functions

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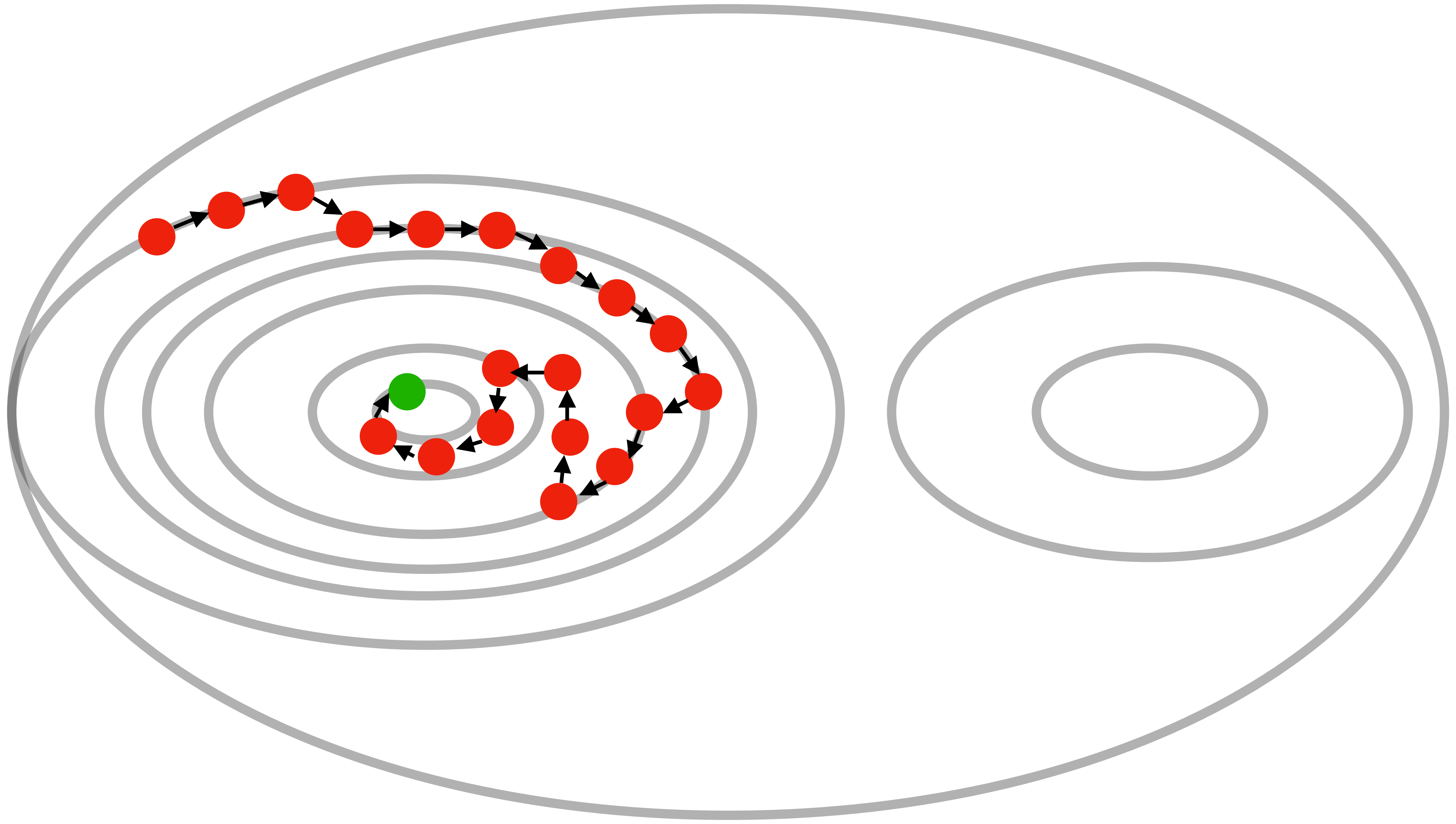

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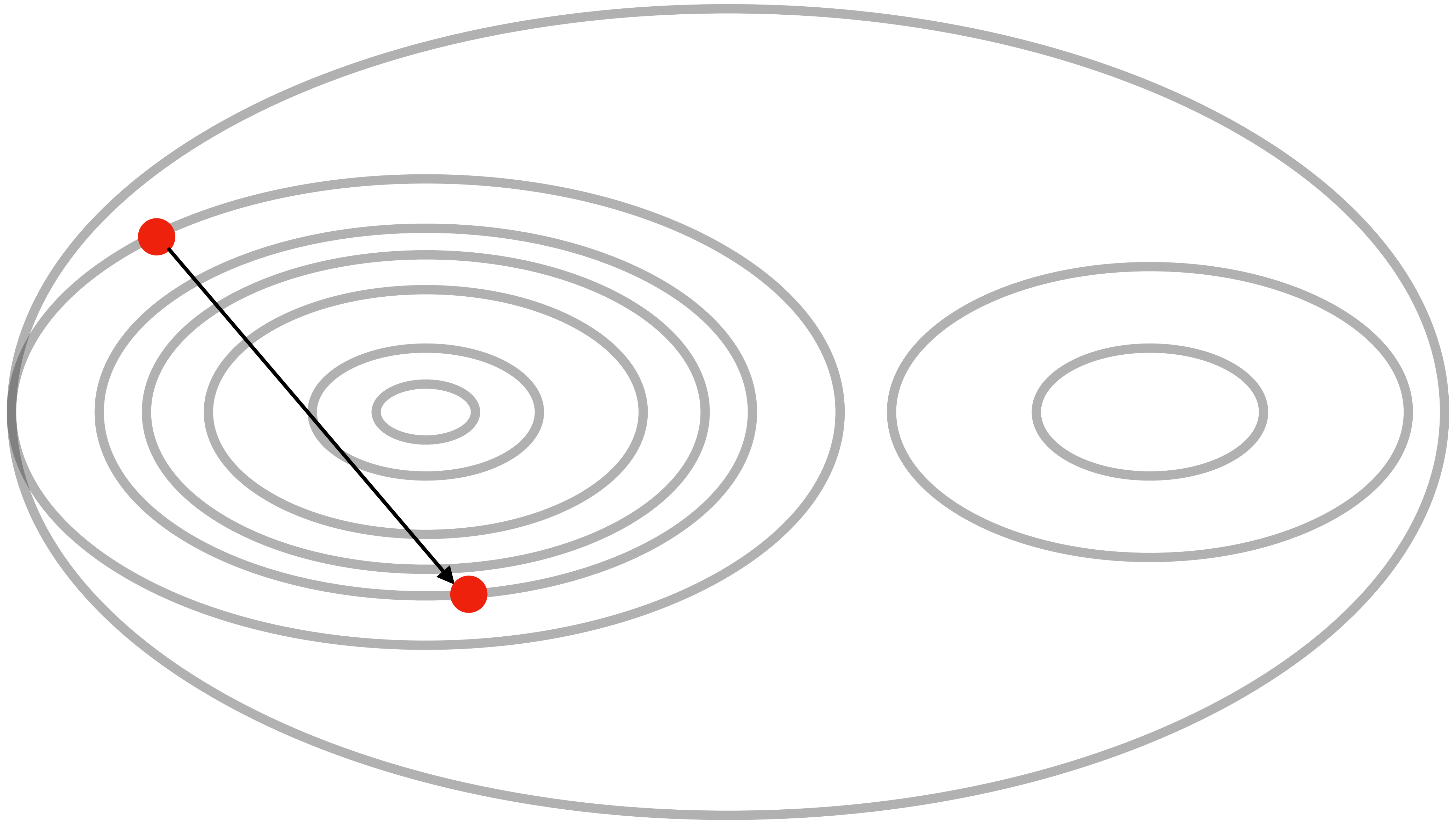

Optimizing Loss Functions

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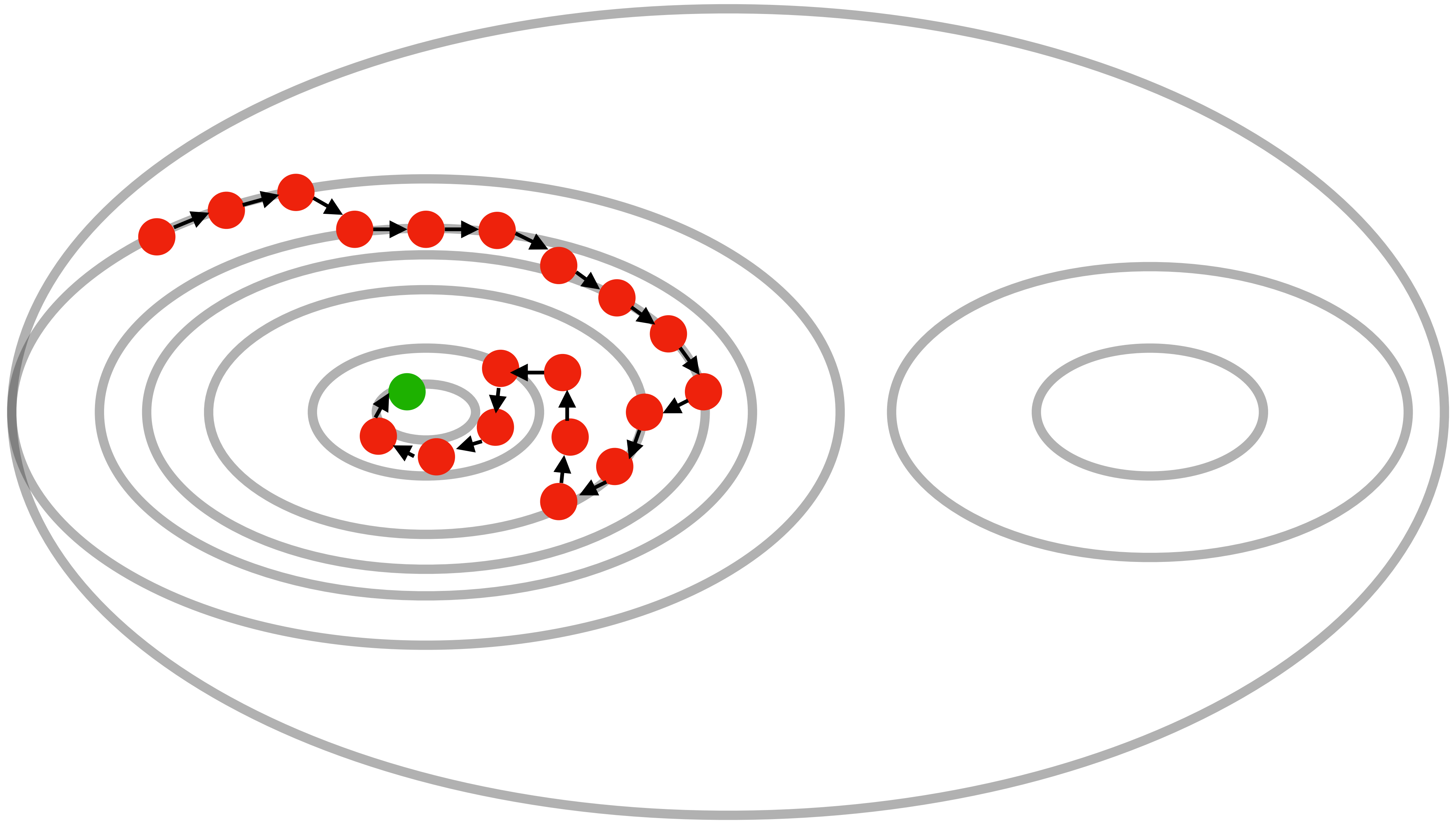

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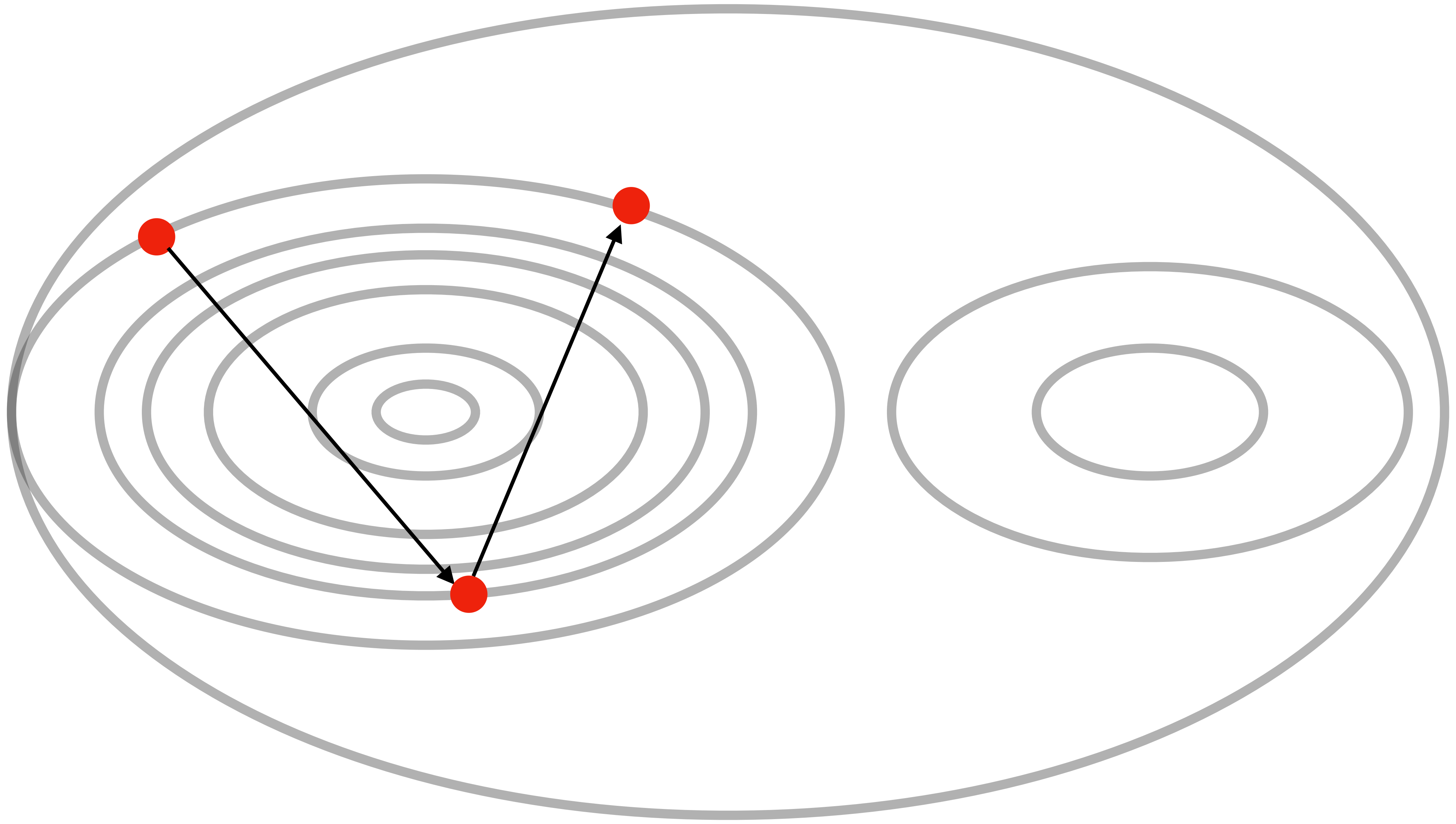

Optimizing Loss Functions

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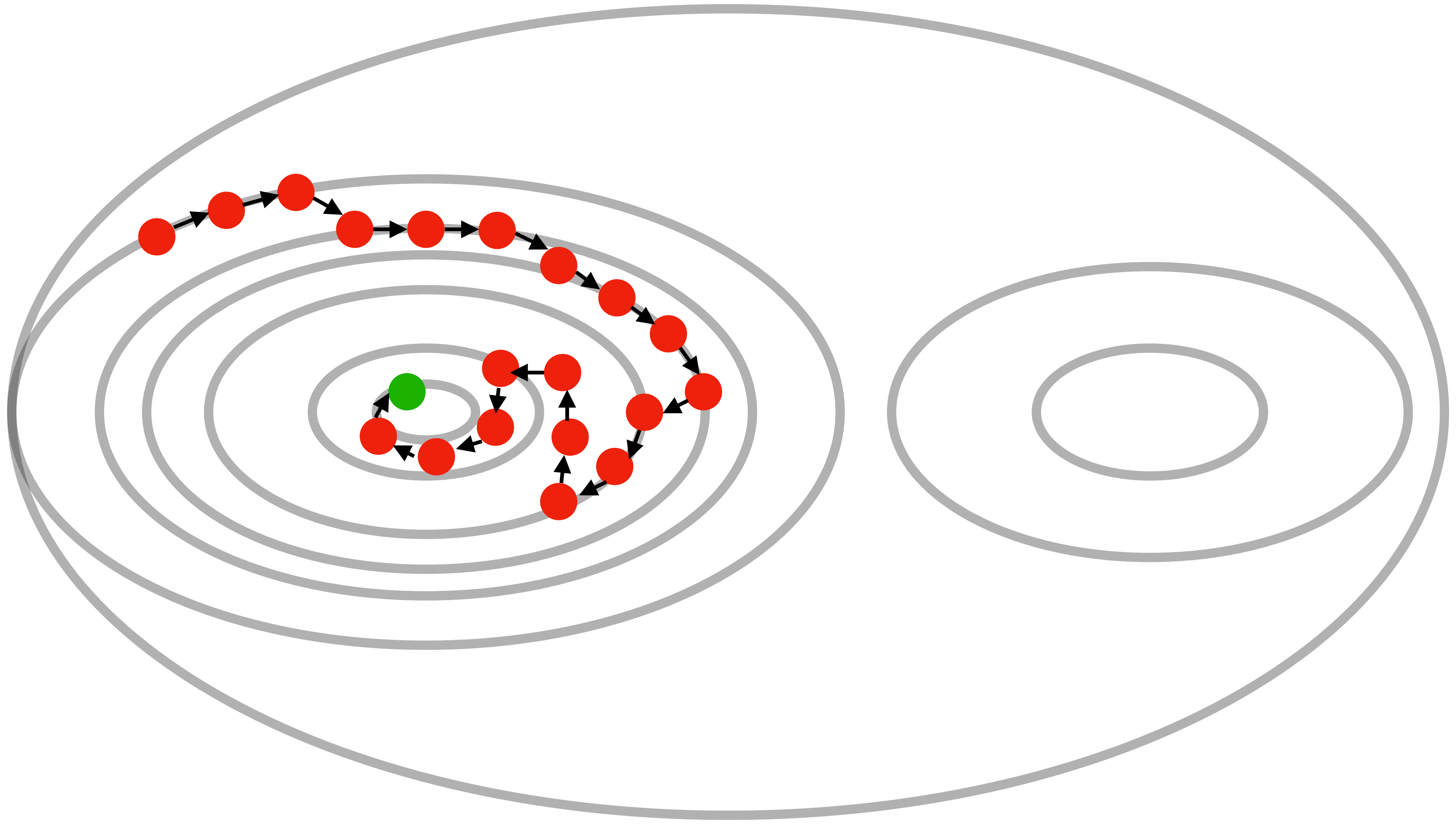

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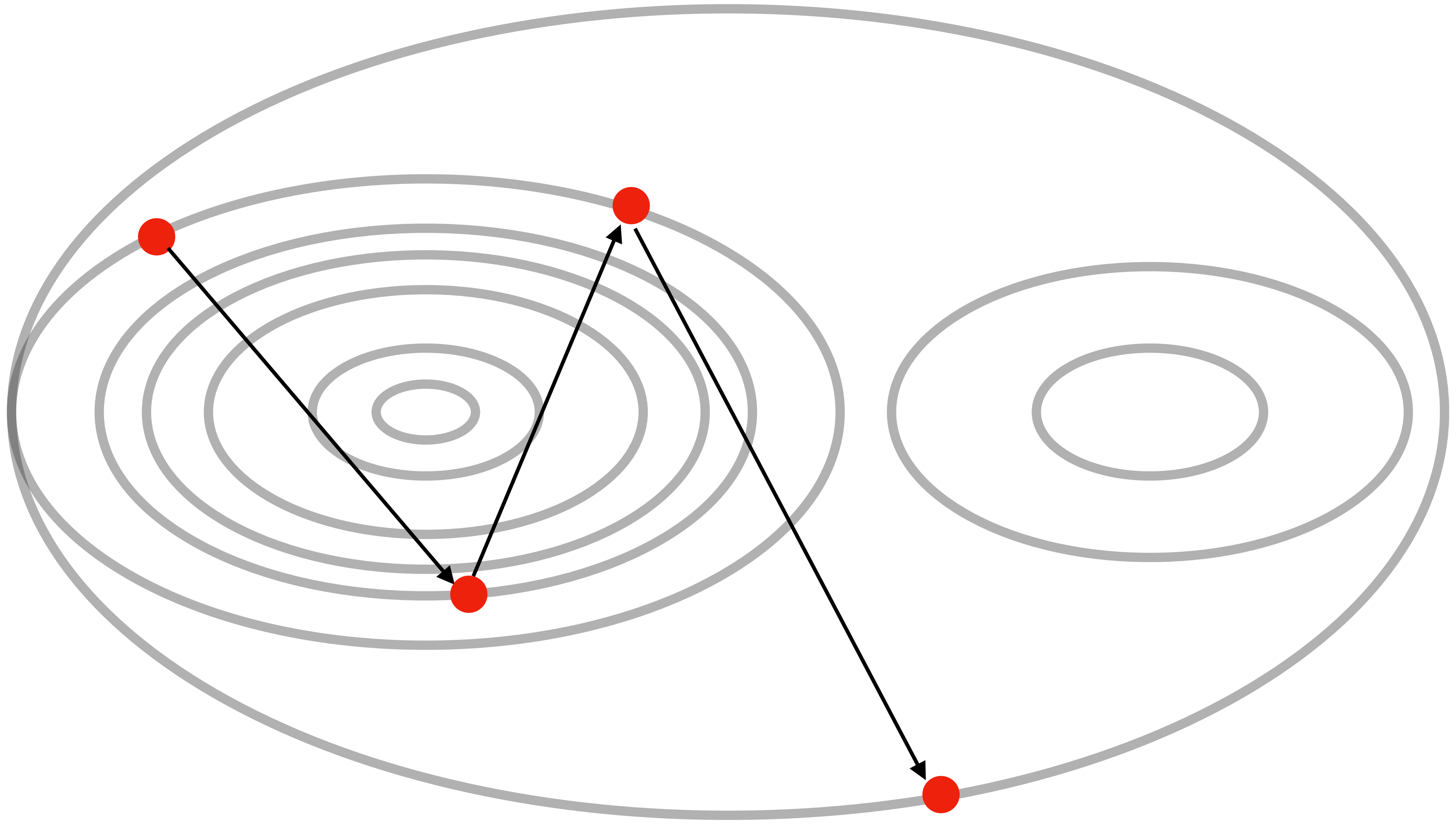
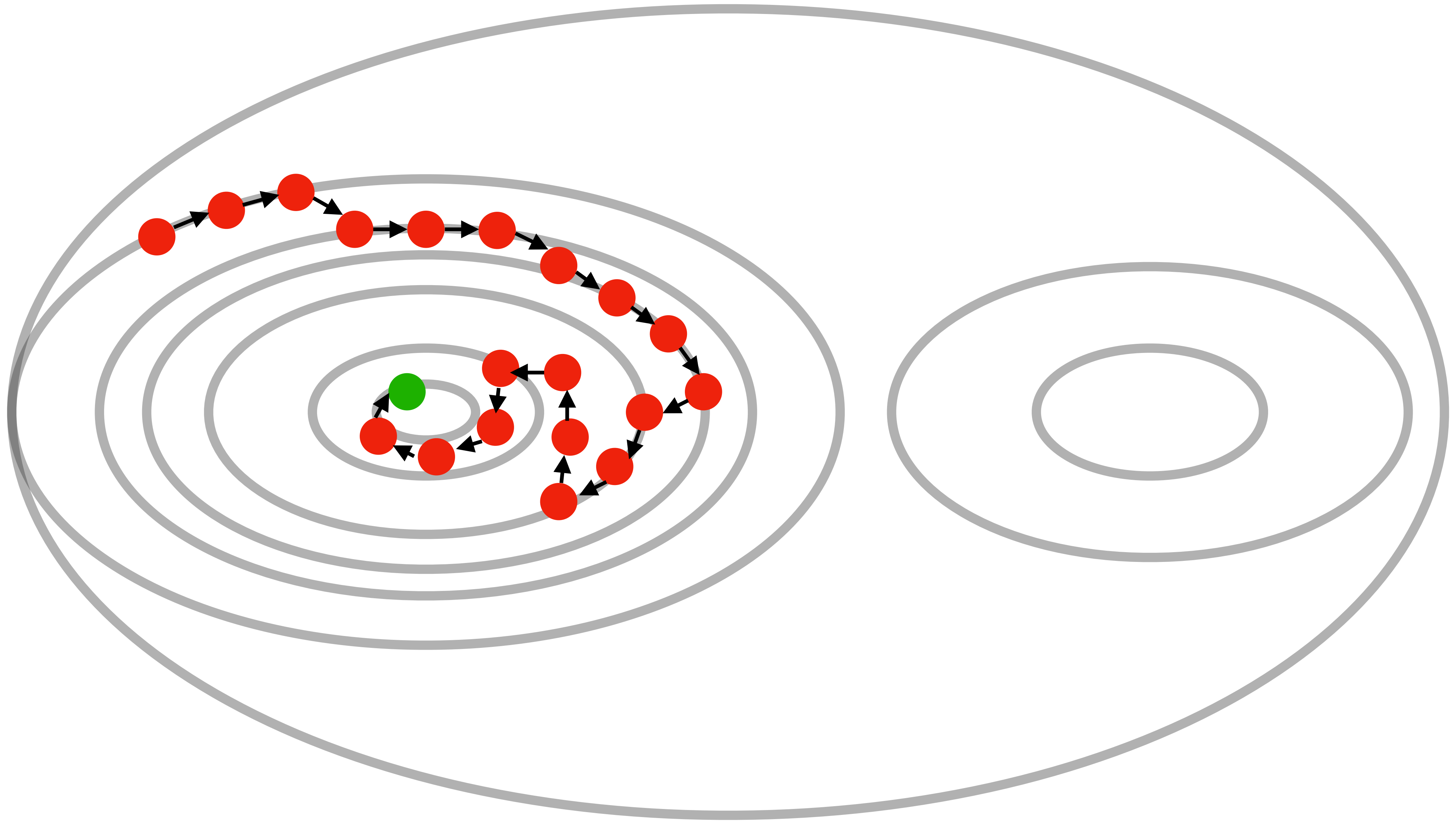


Optimizing Loss Functions

Gradient Descent - Effect of Learning Rate

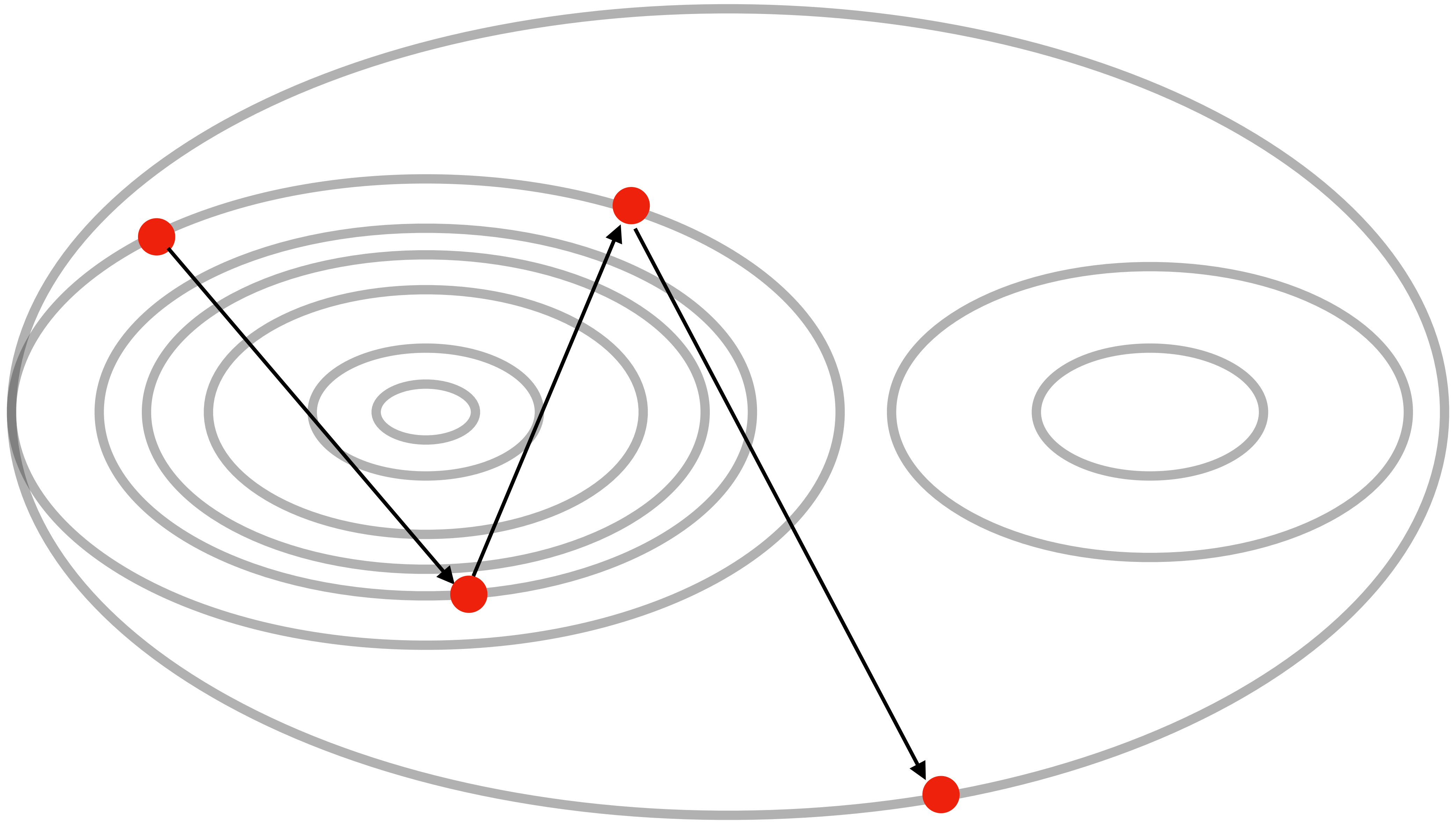
What happens when α is too small?
Say $\alpha = 10^{-5}$

With a small learning rate α, if the loss function is convex, the optimization will eventually **converge**



What happens when α is too large?
Say $\alpha = 10$

With a large learning rate α, if the loss function is convex, the optimization could possibly start **diverging** and never converge



Optimizing Loss Functions

Gradient Descent - Stopping Criterion

Maximum Iteration

Gradient Norm Threshold

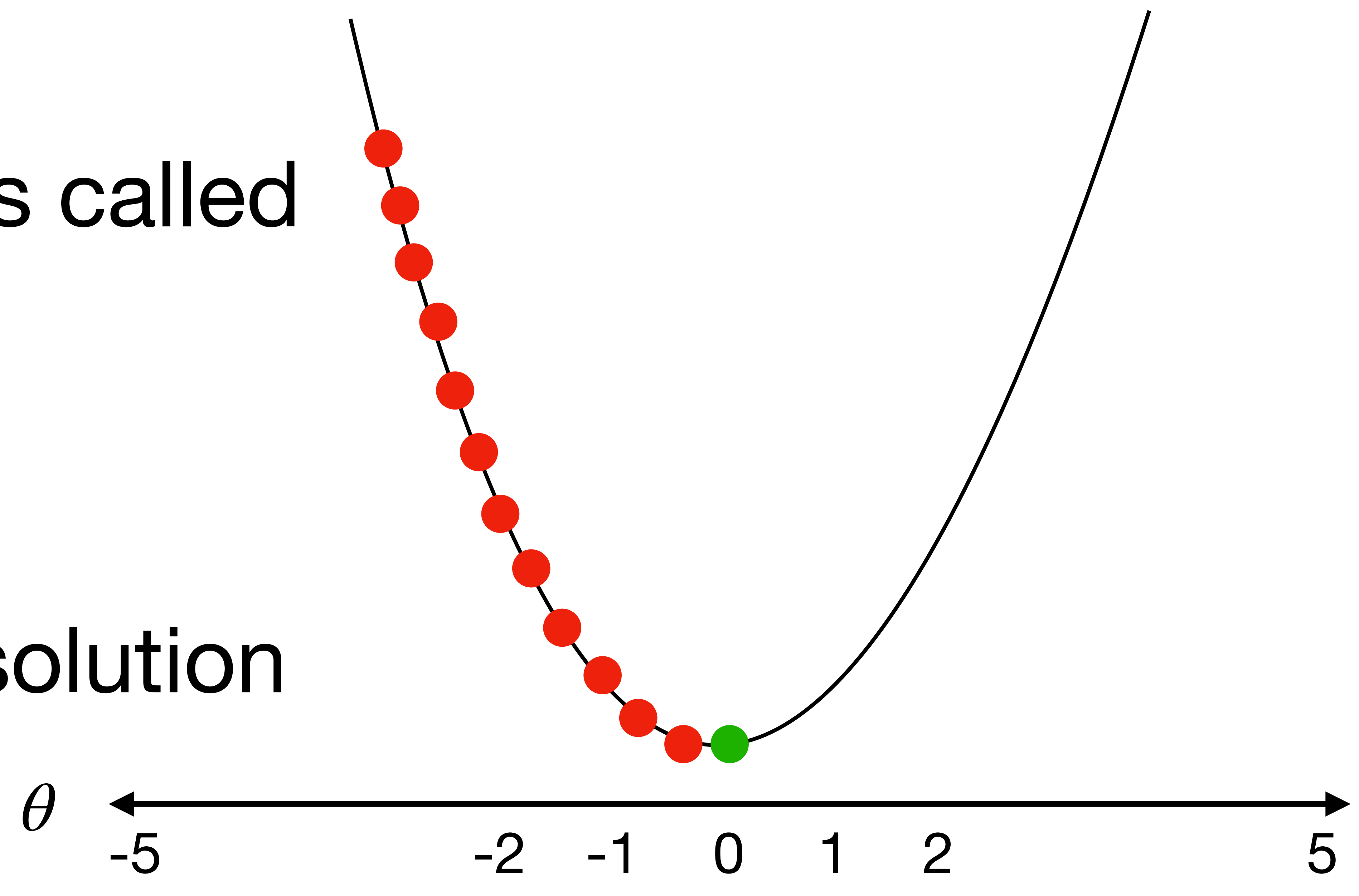
Function Value Change

Parameter Value Change

Optimizing Loss Functions

Gradient Descent - Stopping Criterion

- When do you stop your iterations?
  - Maximum Iteration
    - Each iteration through the training dataset is called an "epoch"
    - Terminate after a fixed number of epochs
    - Simple, but provides no guarantees about solution quality



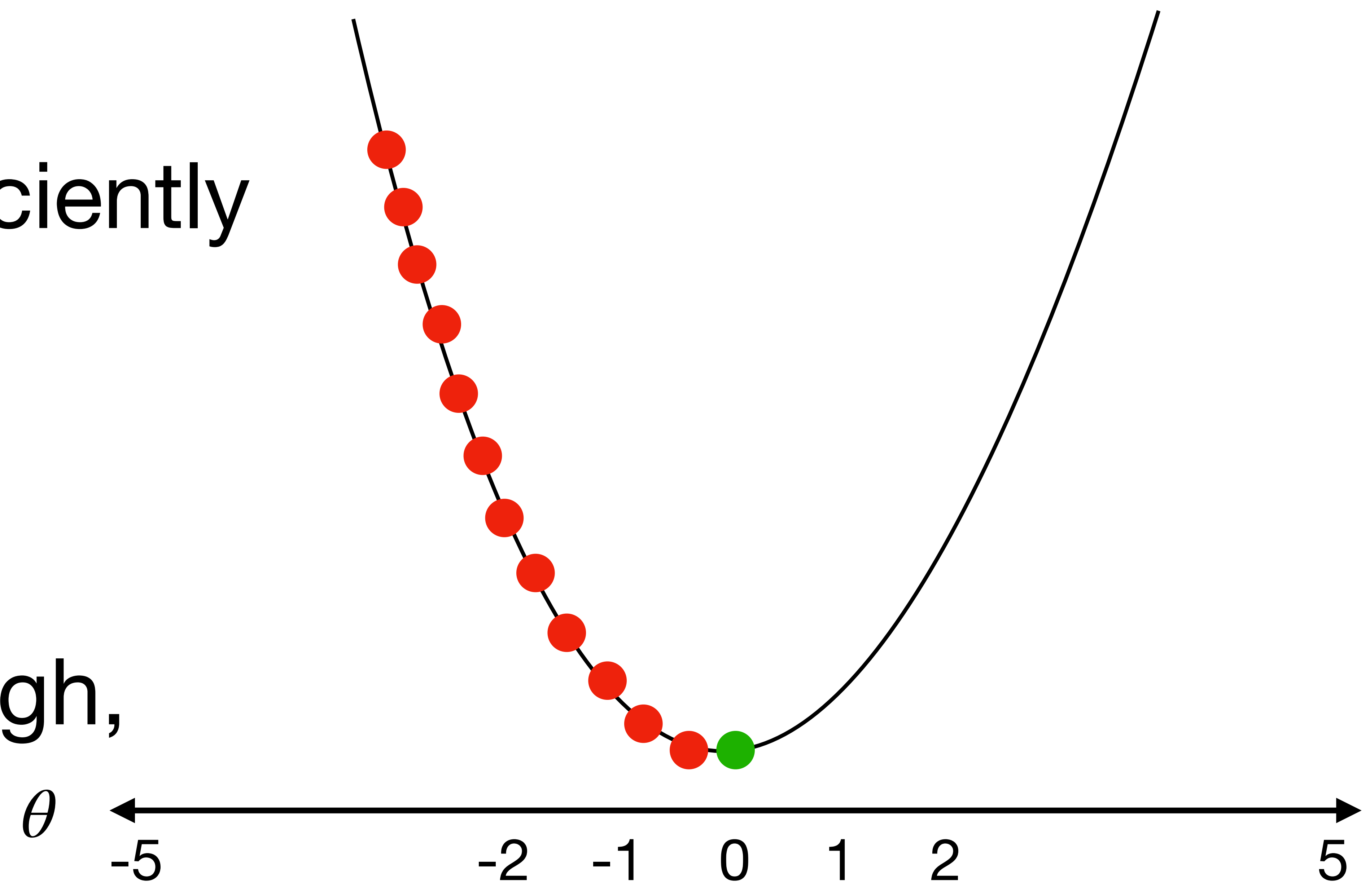
Optimizing Loss Functions

Gradient Descent - Stopping Criterion

- When do you stop your iterations?
  - Gradient Norm Threshold
    - Terminate when the gradient becomes sufficiently small

$$||\nabla \ell_\theta(x)||_2 \leq \epsilon$$

    - At this point, if the gradients are small enough, the parameters won't move much anyway

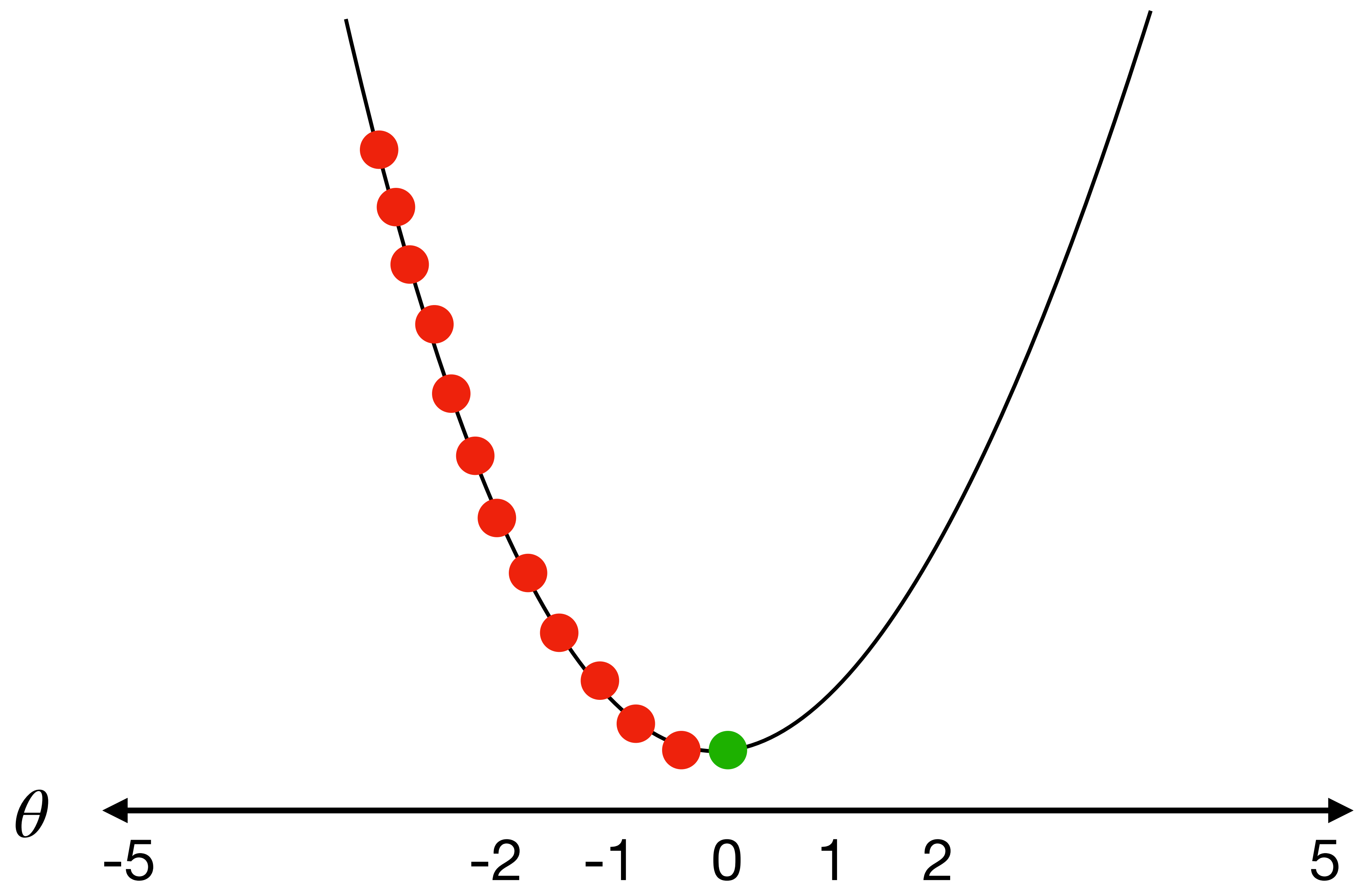


Optimizing Loss Functions

Gradient Descent - Stopping Criterion

- When do you stop your iterations?
  - Function Value Change
    - Terminate when the loss stops changing meaningfully

$$|\ell_{\theta_t}(x) - \ell_{\theta_{t-1}}(x)| \leq \epsilon$$



Optimizing Loss Functions

Gradient Descent - Stopping Criterion

- When do you stop your iterations?
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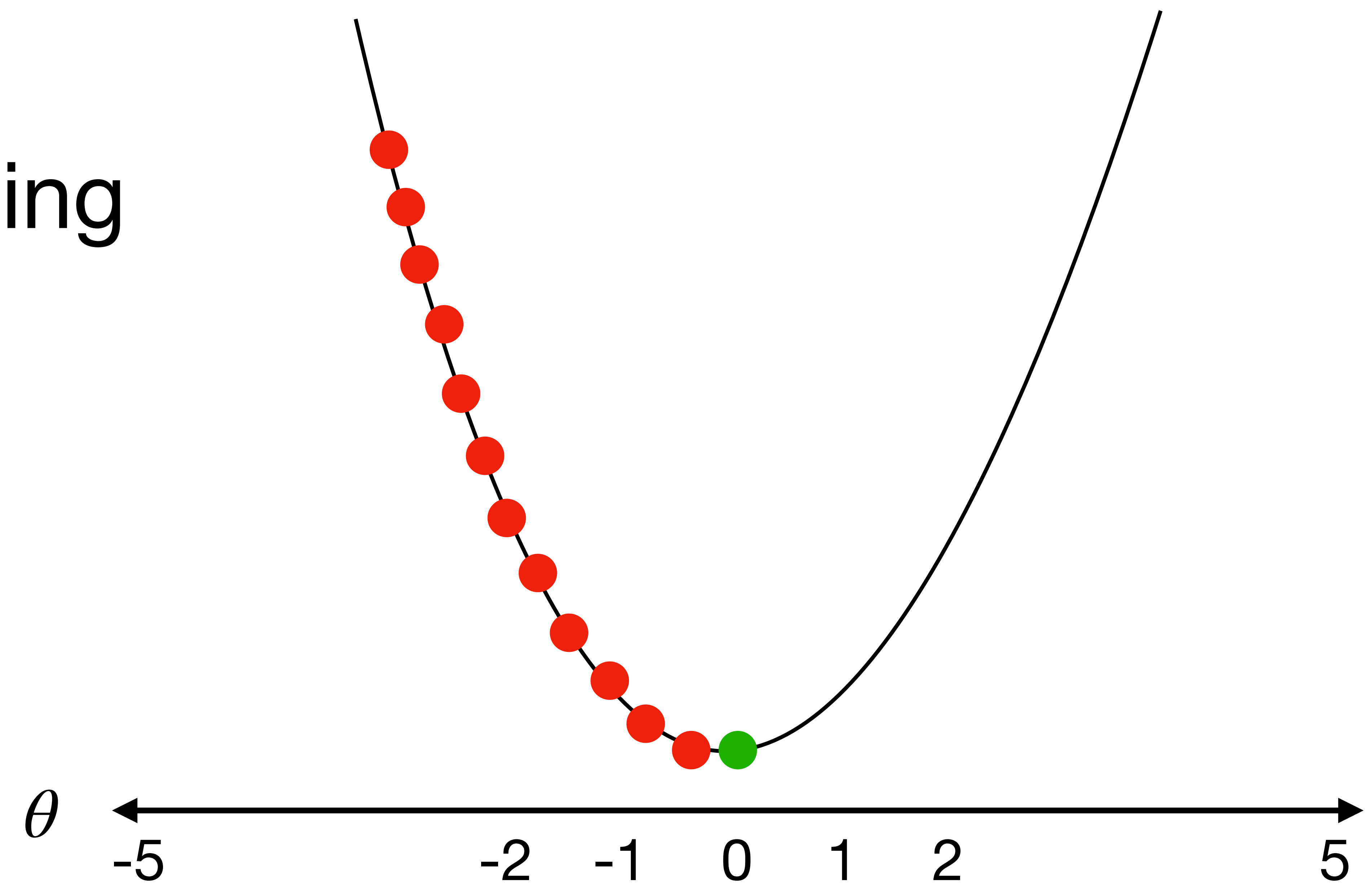
    - Terminate when the parameters stop changing meaningfully

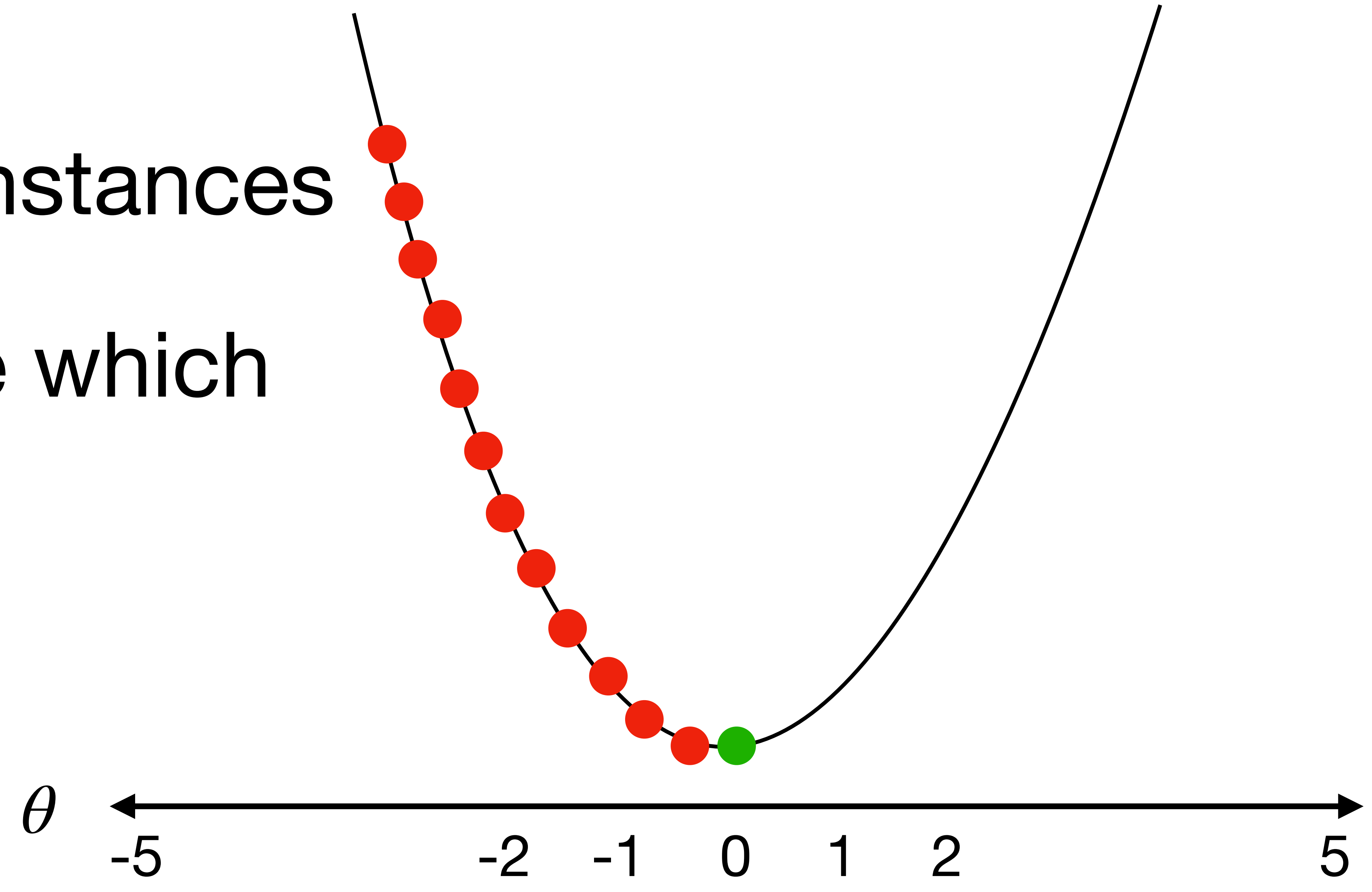
$$|\theta_t - \theta_{t-1}| \leq \epsilon$$



Optimizing Loss Functions

Gradient Descent - Stopping Criterion

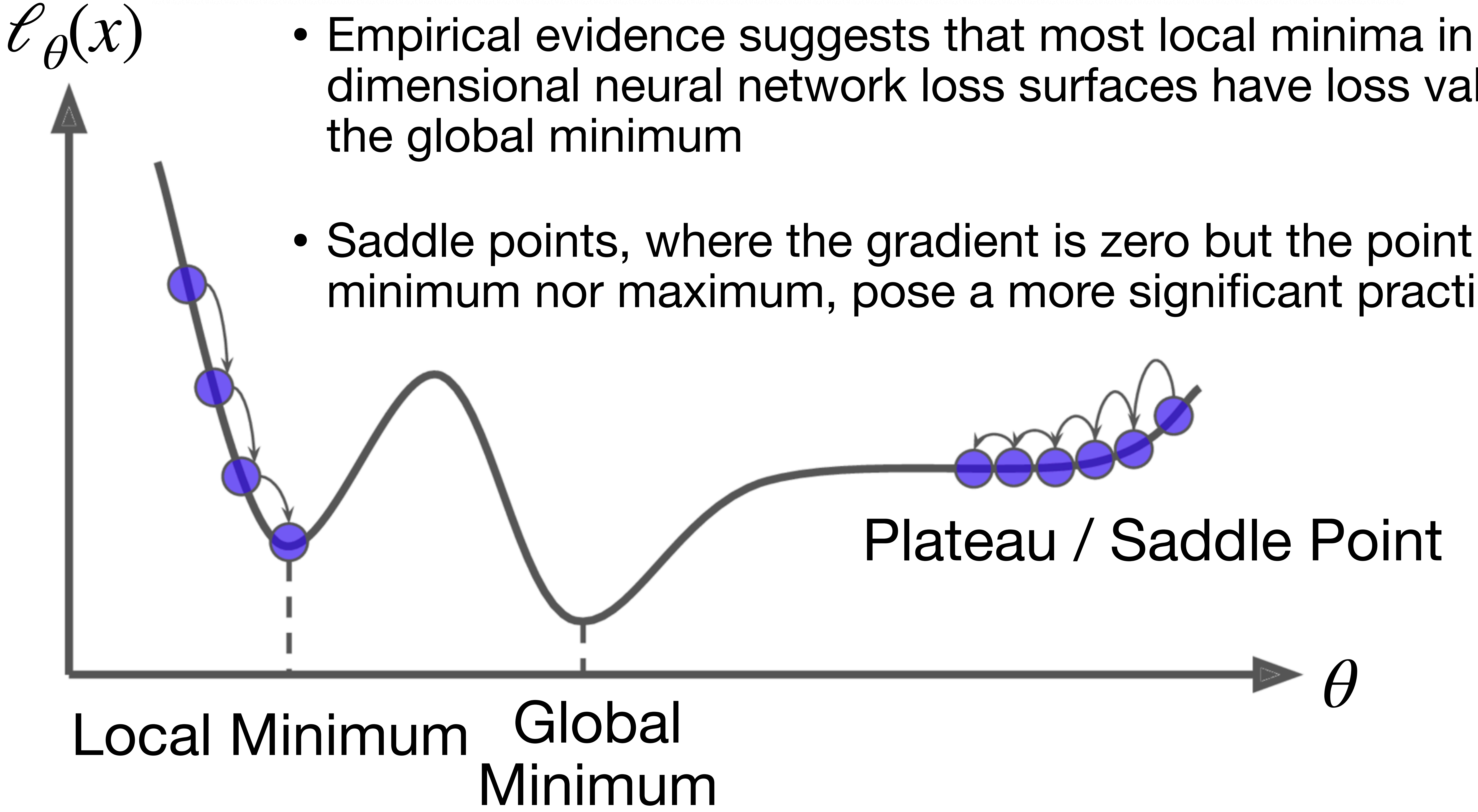
- When do you stop your iterations?
  - Validation Based Stopping (Early Stopping)
    - Monitor performance on a validation set of instances
    - Stop when validation loss begins to increase which signals overfitting
    - Serves as both stopping criterion and regularization



Optimizing Loss Functions

Gradient Descent - More Complicated Functions

- Most deep learning models however have **highly non-convex** loss landscapes
- Empirical evidence suggests that most local minima in high-dimensional neural network loss surfaces have loss values close to the global minimum
- Saddle points, where the gradient is zero but the point is neither a minimum nor maximum, pose a more significant practical challenge.



Optimizing Loss Functions

Gradient Descent - Momentum

Optimizing Loss Functions

Gradient Descent - Momentum

Optimizing Loss Functions

Gradient Descent - Momentum

- Standard gradient descent can oscillate in ravines
    - Areas where the surface curves **more steeply in one dimension** than another
    - Or they can get stuck in plateau / saddle points
- Momentum helps accelerate gradient descent by accumulating velocity in **directions of consistent gradient**

$$\theta_j \leftarrow \theta_j - \alpha \cdot \frac{\partial \ell_\theta(x)}{\partial \theta_j}$$

$$\theta_t = \theta_{t-1} - \alpha \nabla \ell_{\theta_{t-1}}$$

Optimizing Loss Functions

Gradient Descent - Momentum

- Momentum helps accelerate gradient descent by accumulating velocity in **directions of consistent gradient**

$$\theta_t = \theta_{t-1} - \alpha \nabla \ell_{\theta_{t-1}}$$

With Momentum

$$v_t = \beta \cdot v_{t-1} + \nabla \ell_{\theta_{t-1}}$$

$$\theta_t = \theta_{t-1} - \alpha \cdot v_t$$

Optimizing Loss Functions

Gradient Descent - Momentum

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$$\theta_t = \theta_{t-1} - \alpha \nabla \ell_{\theta_{t-1}}$$

With Momentum

Velocity Vector $\boxed{v_t} = \beta \cdot v_{t-1} + \nabla \ell_{\theta_{t-1}}$

$$\theta_t = \theta_{t-1} - \alpha \cdot \boxed{v_t}$$

Optimizing Loss Functions

Gradient Descent - Momentum

- Momentum helps accelerate gradient descent by accumulating velocity in **directions of consistent gradient**

$$\theta_t = \theta_{t-1} - \alpha \nabla \ell_{\theta_{t-1}}$$

With Momentum

β is the momentum coefficient, typically set to 0.9

$$v_t = \beta \cdot v_{t-1} + \nabla \ell_{\theta_{t-1}}$$

$$\theta_t = \theta_{t-1} - \alpha \cdot v_t$$

Optimizing Loss Functions

Gradient Descent - Momentum

- Momentum helps accelerate gradient descent by accumulating velocity in **directions of consistent gradient**

$$\theta_t = \theta_{t-1} - \alpha \nabla \ell_{\theta_{t-1}}$$

With Momentum

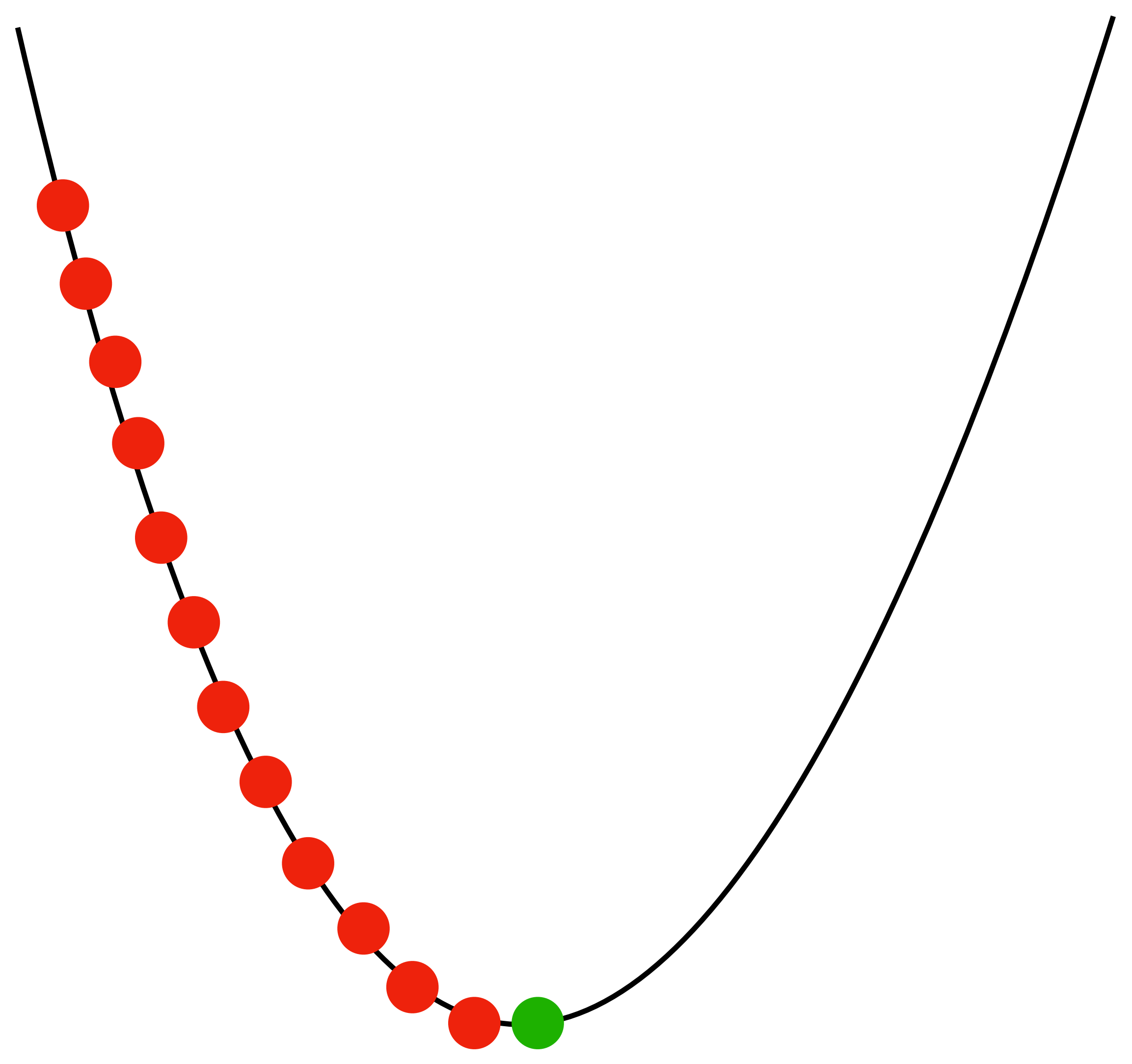
If $\beta = 0$, you get back standard gradient descent

$$v_t = \boxed{\beta} \cdot v_{t-1} + \nabla \ell_{\theta_{t-1}}$$

$$\theta_t = \theta_{t-1} - \alpha \cdot v_t$$

Optimizing Loss Functions

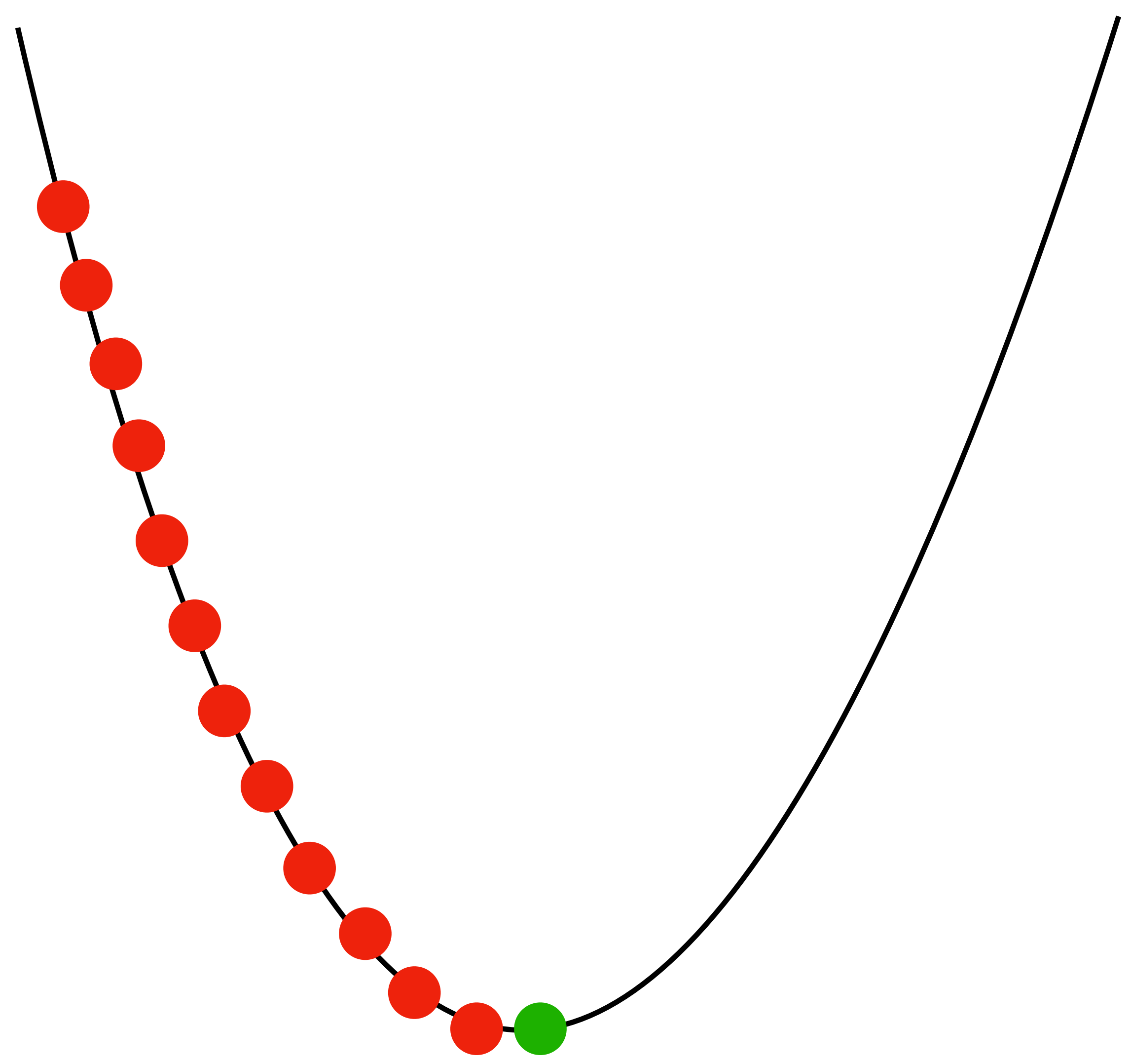
Gradient Descent - Adaptive Step Sizes



α is too small
Finds the optimal but too slow

Optimizing Loss Functions

Gradient Descent - Adaptive Step Sizes



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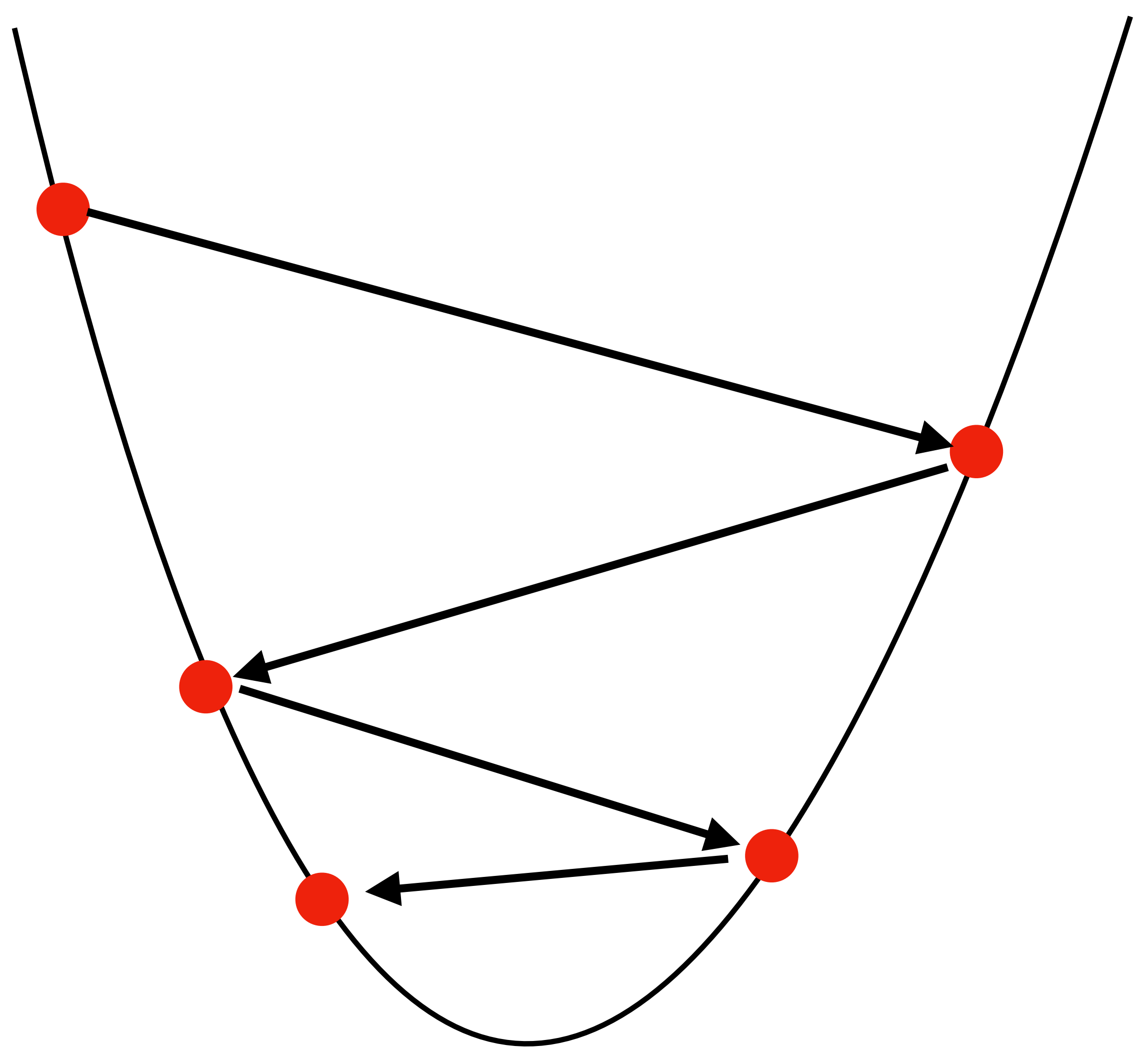
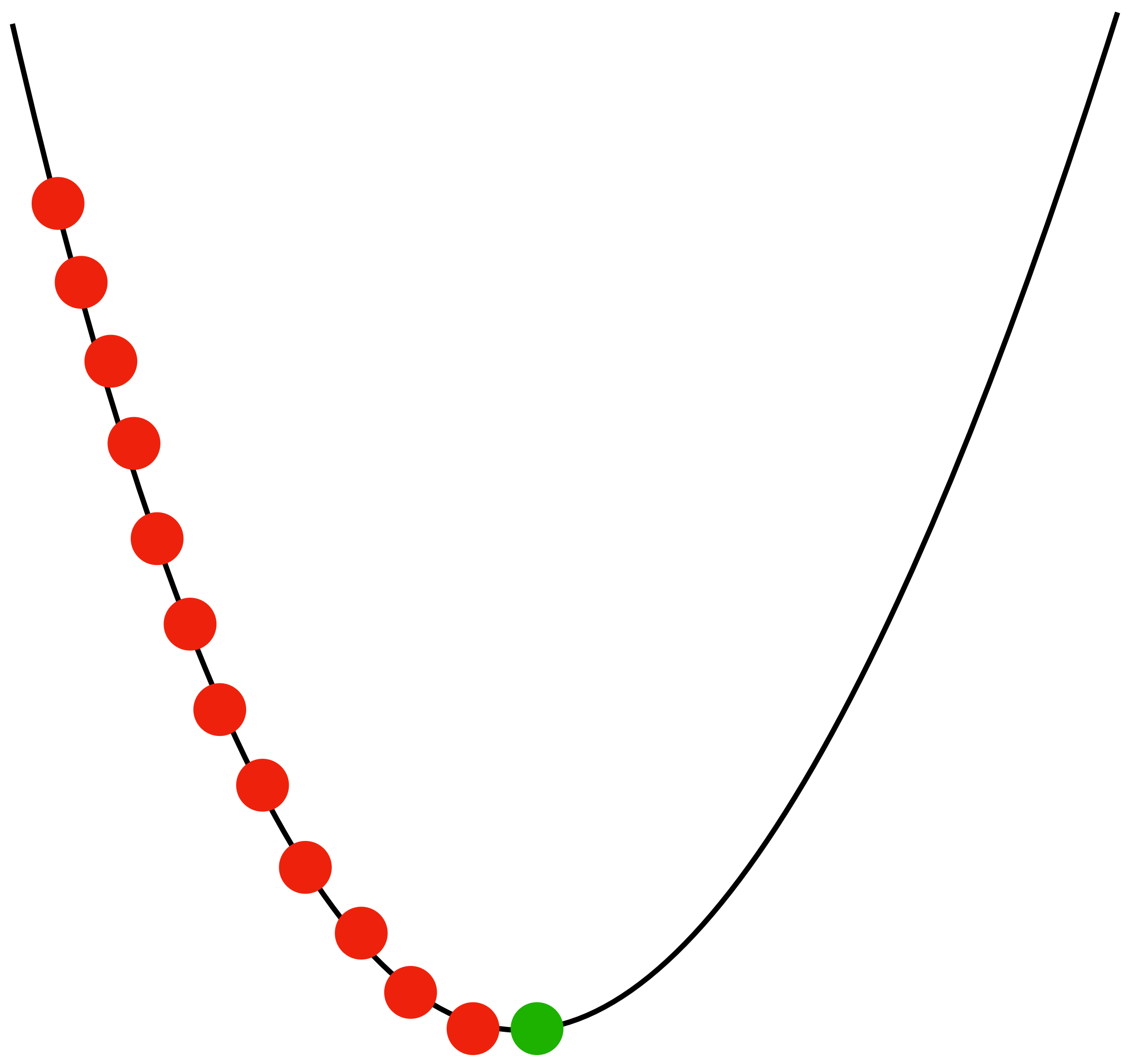


α is too large
Might not find optimal
Could even begin to diverge

Optimizing Loss Functions

Gradient Descent - Adaptive Step Sizes



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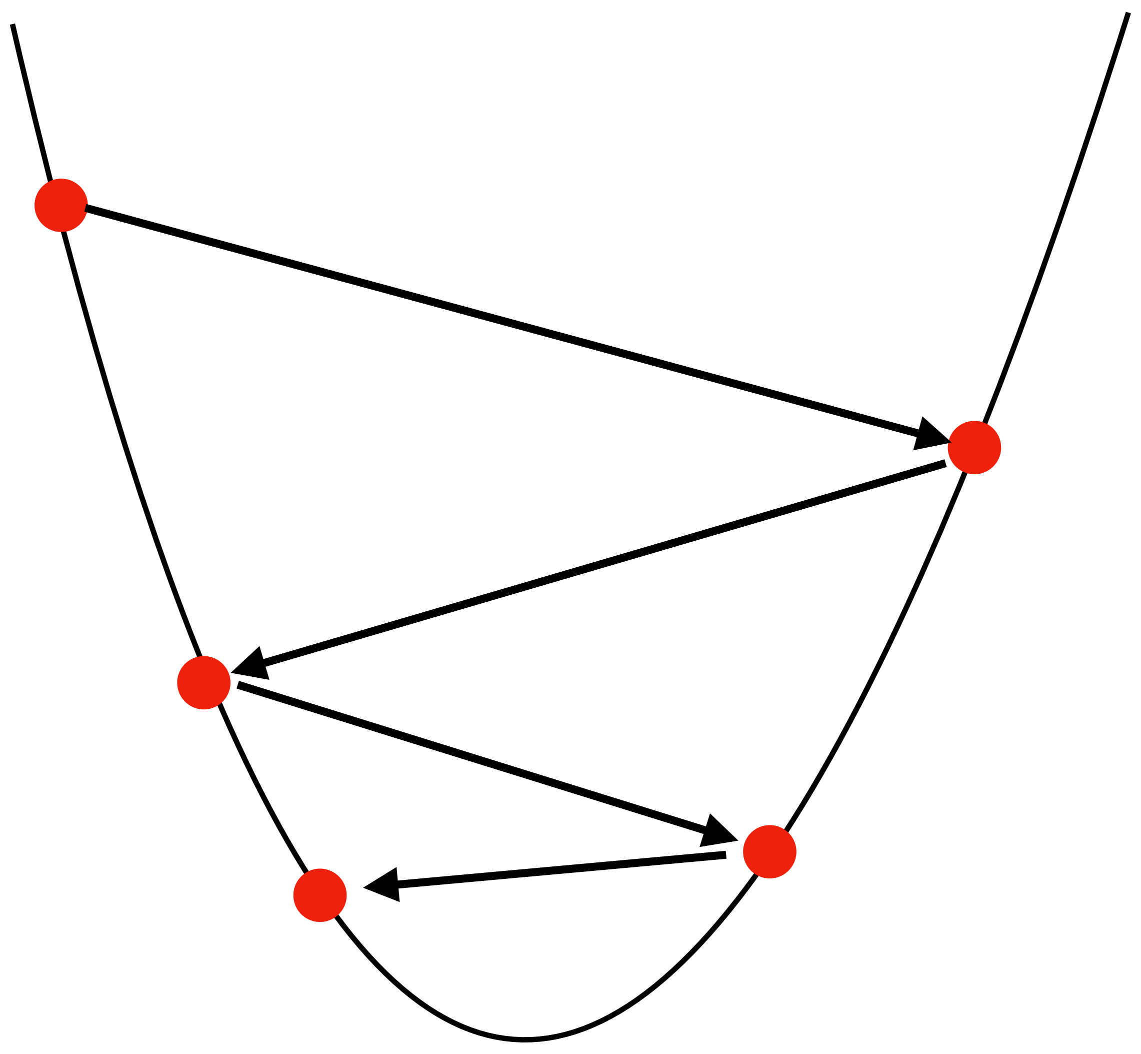

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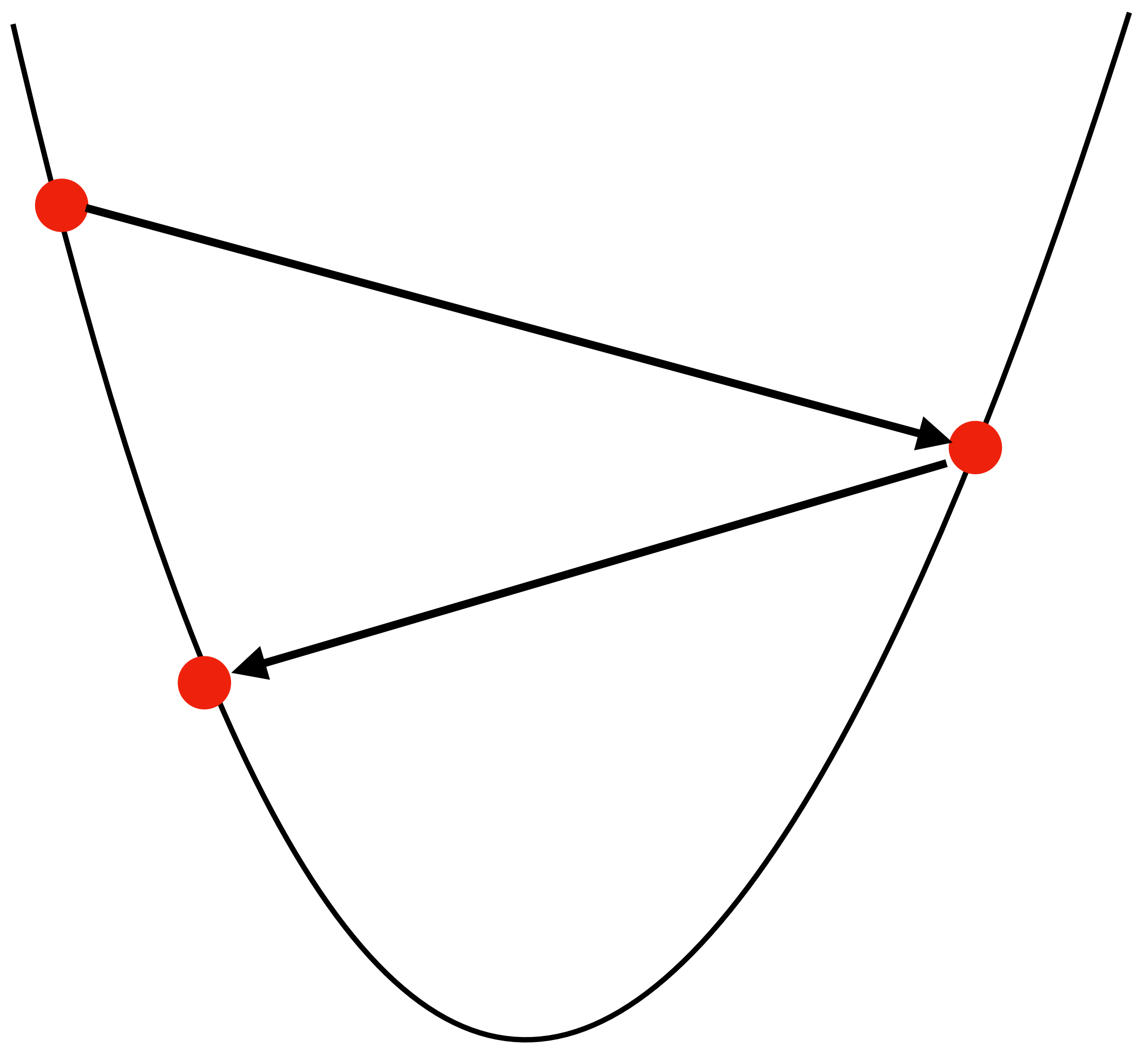
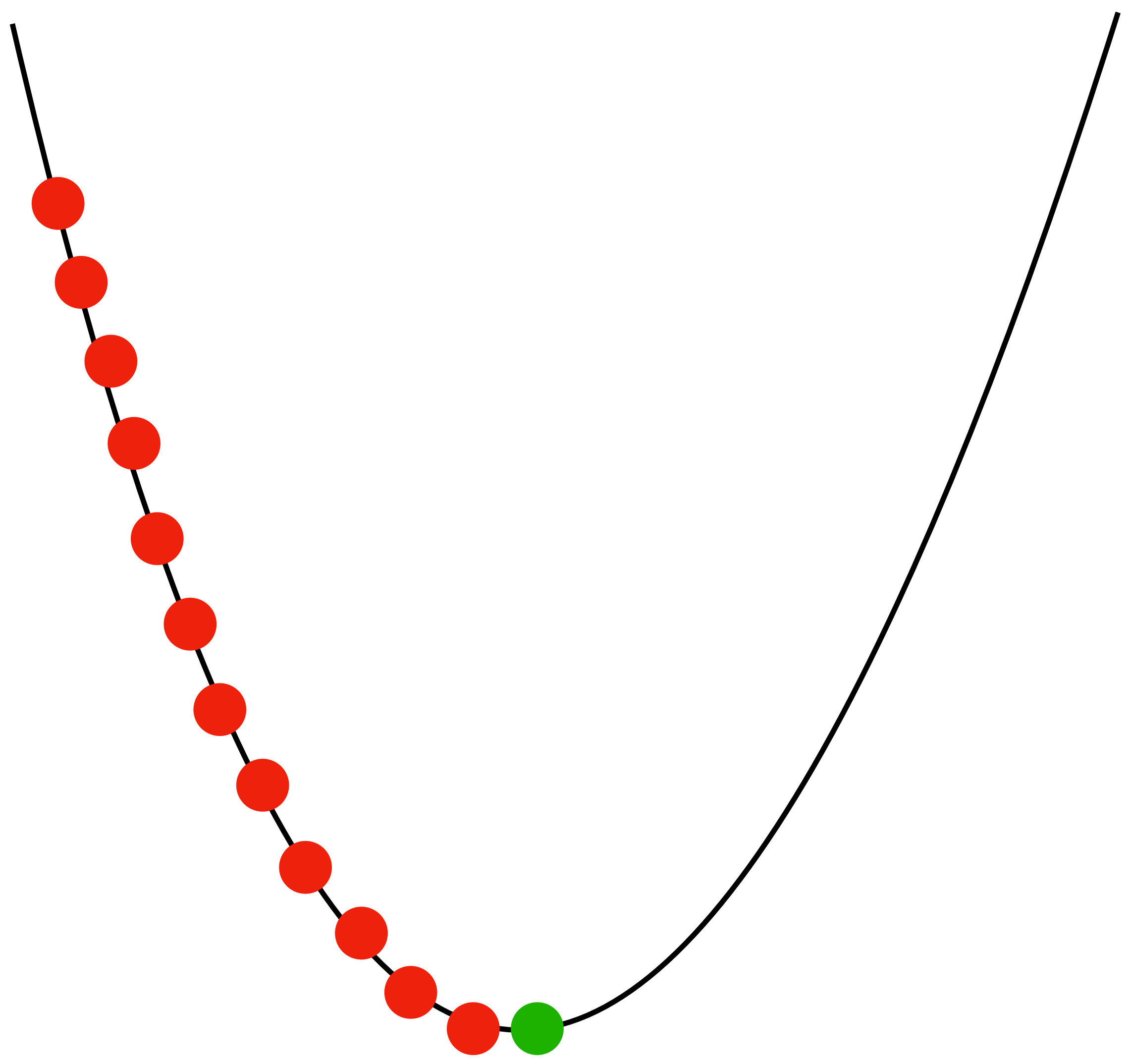


What if you set α to be large initially?

Optimizing Loss Functions

Gradient Descent - Adaptive Step Sizes



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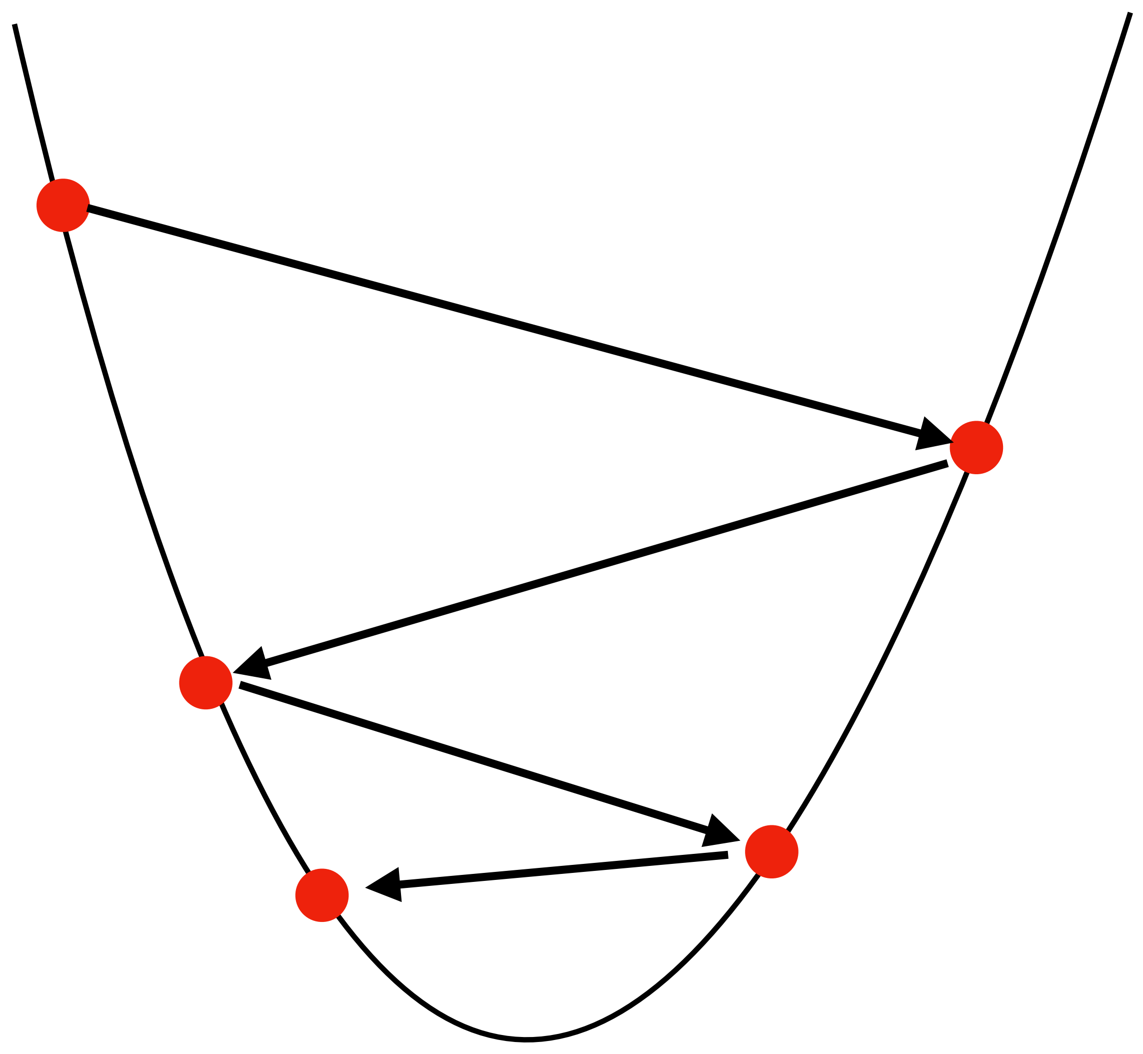

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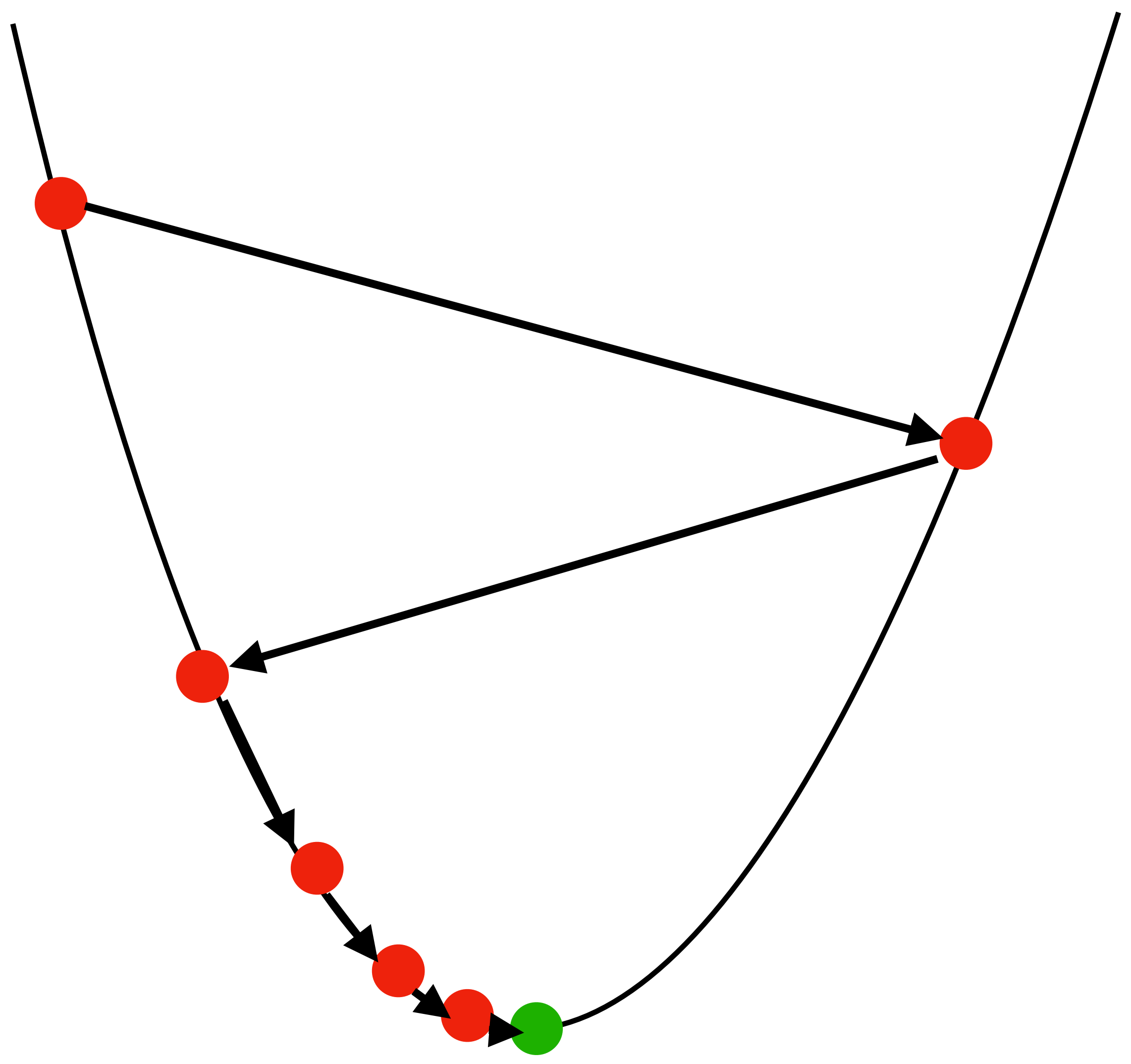

And keep reducing α as
number of epochs increases?

Optimizing Loss Functions

Gradient Descent - Per Parameter Adaptive Learning Rates

- A single global learning rate may be suboptimal
  - Some parameters might benefit from larger updates while others need smaller ones.
  - Adaptive methods adjust the learning rate for each parameter individually based on historical gradient information.

Optimizing Loss Functions
Gradient Descent - AdaGrad

Optimizing Loss Functions

Gradient Descent - AdaGrad + RMSProp

Optimizing Loss Functions

Gradient Descent - AdaGrad

- AdaGrad adapts the learning rate for **each parameter** based on the sum of squared historical gradients

$$G_t = G_{t-1} + (\nabla \ell_{\theta_{t-1}})^2$$

$$\theta_t = \theta_{t-1} - \frac{\alpha}{\sqrt{G_t} + \epsilon} \cdot \nabla \ell_{\theta_{t-1}}$$

  - Parameters with <u>large</u> historical gradients receive <u>smaller</u> updates
  - Parameters with <u>small</u> historical gradients receive <u>larger</u> updates
  - The limitation is that the accumulated sum G_t grows monotonically, eventually making the learning rate vanishingly small.

Optimizing Loss Functions

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  - **The limitation is that the accumulated sum G_t grows monotonically, eventually making the learning rate vanishingly small.**

Optimizing Loss Functions

Gradient Descent - RMSProp

- RMSprop addresses AdaGrad's diminishing learning rate by using an exponentially decaying average of squared gradients

$$G_t = \rho \cdot G_{t-1} + (1 - \rho) \cdot (\nabla \ell_{\theta t-1})^2$$

$$\theta_t = \theta_{t-1} - \frac{\alpha}{\sqrt{G_t} + \epsilon} \cdot \nabla \ell_{\theta_{t-1}}$$

  - The decay rate ρ is typically set to 0.9.
  - This prevents the learning rate from decaying to zero while still adapting to the gradient scale.

Optimizing Loss Functions

Gradient Descent - ADAM

- Adam (**Ada**ptive **M**oment Estimation) combines the benefits of **momentum** (first moment) with the adaptive learning rates of **RMSProp** (second moment)

Optimizing Loss Functions

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Optimizing Loss Functions

Gradient Descent - ADAM

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Adam maintains **two** moving averages

First Moment (mean): $m_t = \beta_1 m_{t-1} + (1 - \beta_1) \nabla \ell_{\theta_{t-1}}$

Second Moment (variance): $v_t = \beta_2 v_{t-1} + (1 - \beta_2)(\nabla \ell_{\theta_{t-1}})^2$

Optimizing Loss Functions
Gradient Descent - ADAM

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Update: $\theta_t = \theta_{t-1} - \dfrac{\alpha}{\sqrt{v_t} + \epsilon} \cdot m_t$

Optimizing Loss Functions

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Update: $\theta_t = \theta_{t-1} - \dfrac{\alpha}{\sqrt{\hat{v}_t} + \epsilon} \cdot \hat{m}_t$

Bias Correction: Important for early iterations when estimates are biased towards 0

$$\hat{m}_t = \frac{m_t}{1 - \beta_1^t}$$

$$\hat{v}_t = \frac{v_t}{1 - \beta_2^t}$$

Optimizing Loss Functions

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Default Hyperparameters: $\beta_1 = 0.9,\ \beta_2 = 0.999,\ \alpha = 10^{-3}$

Bias Correction: Important for early iterations when estimates are biased towards 0

$$\hat{m}_t = \frac{m_t}{1 - \beta_1^t}$$

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Gradient Descent

Batch vs Mini-Batch vs Stochastic Gradient Descent

Gradient Descent

Batch vs Mini-Batch vs Stochastic Gradient Descent

- Batch Gradient Descent
  - Use **entire training set per epoch**
  - The whole training dataset is used to compute a single parameter update

$$\theta_t = \theta_{t-1} - \alpha \frac{1}{m} \sum_{i=1}^{m} \nabla \ell_{\theta_{t-1}}(x_i, y_i)$$

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Batch vs Mini-Batch vs Stochastic Gradient Descent

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    - One epoch leads to **one** parameter update

$$\theta_t = \theta_{t-1} - \alpha \frac{1}{m} \boxed{\sum_{i=1}^{m}} \nabla \ell_{\theta_{t-1}}(x_i, y_i)$$

Sum over the whole training dataset

Gradient Descent

Batch vs Mini-Batch vs Stochastic Gradient Descent

- Stochastic Gradient Descent
  - Use **one** randomly selected training data point at each step
  - Parameters are updated after looking at each data point
  - One epoch leads to **m** parameter updates

$$\theta_t = \theta_{t-1} - \alpha \nabla \ell_{\theta_{t-1}}(x_i, y_i)$$

Train / Test Splits

- Generally data is split into a training dataset and a testing data
- Rough rule of thumb is that this is an 80-20 split

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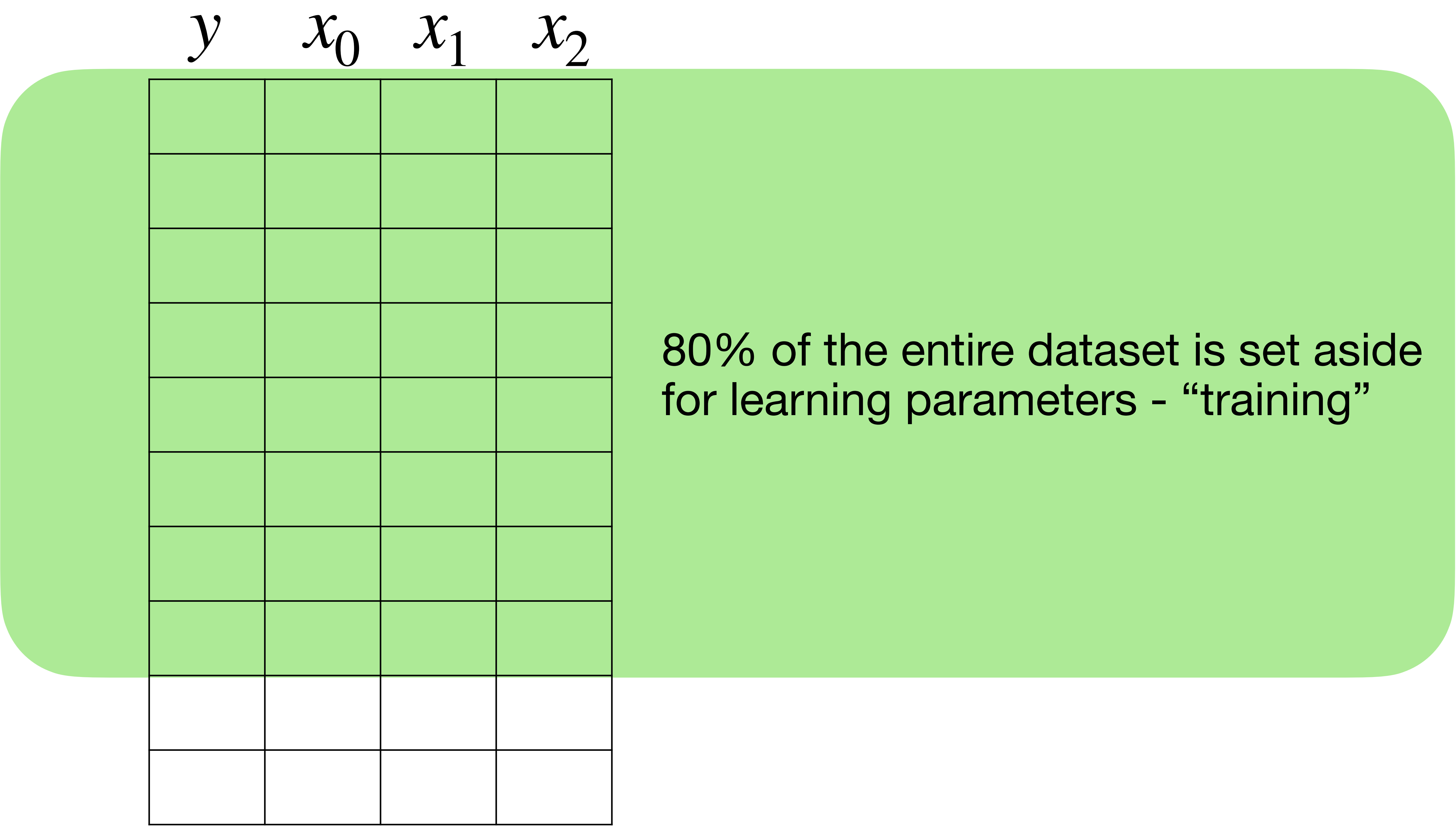

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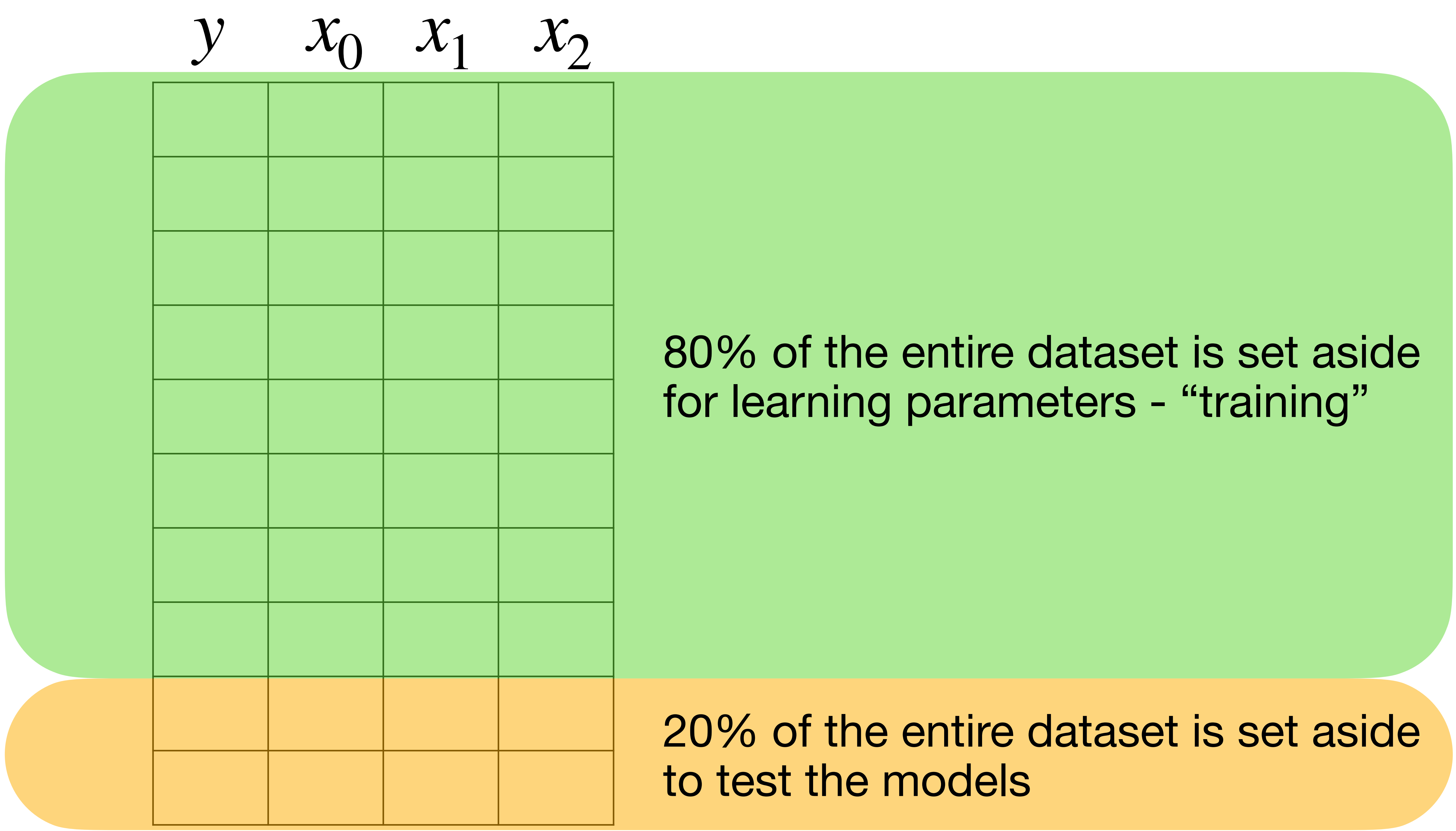


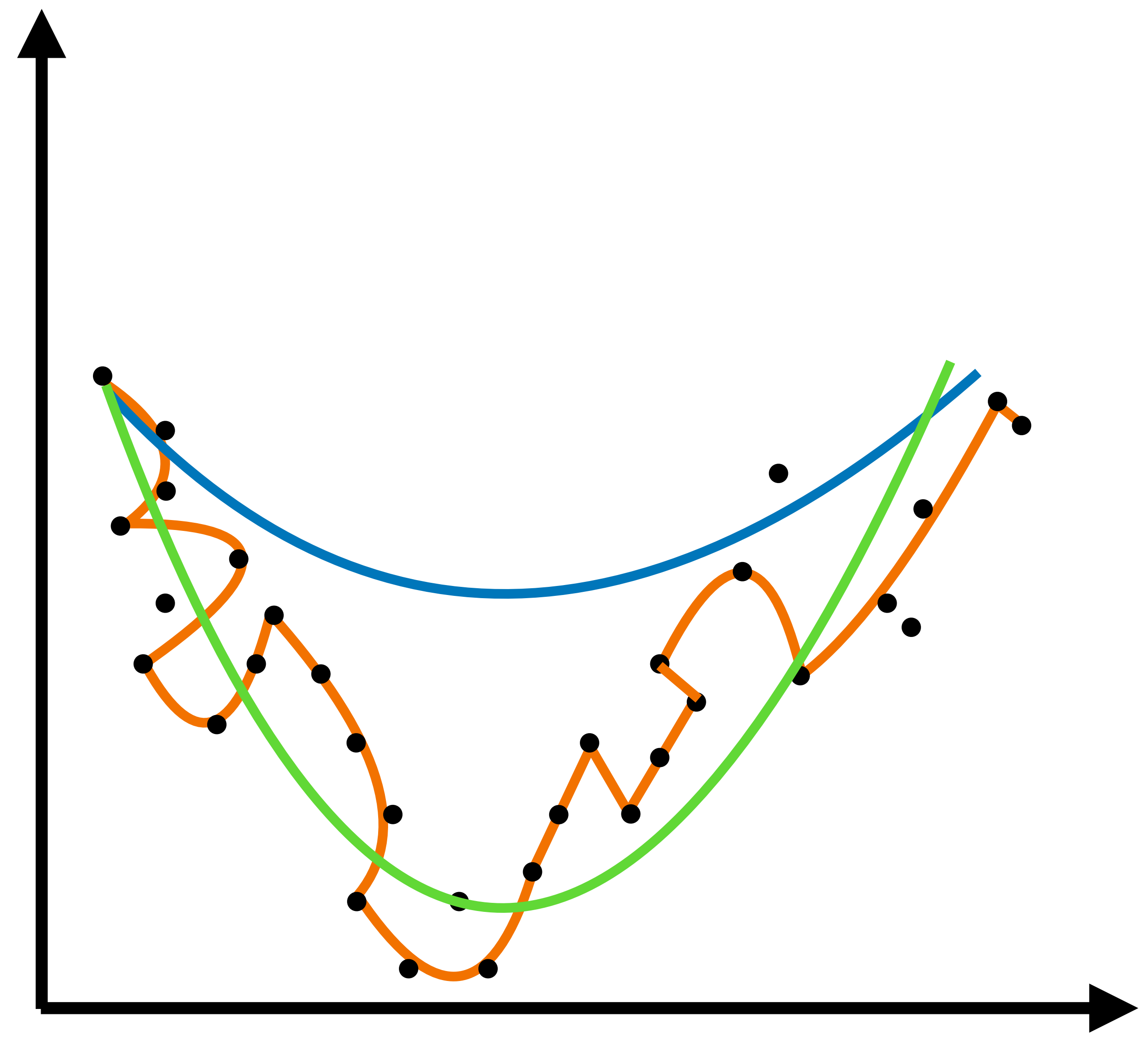
This is **unseen** data and tells you if the model can generalize well

Train / Test Splits

- However, in practice, if you are given only one train and test set, its easy to accidentally pick model architectures that work well on the test set, even though test set data is unseen
- To counter this, we use two unseen datasets - “validation” set and “test” set
- The split is generally of the form 80-10-10 where 80% is training data, 10% is validation data and 10% is test data

Practical Issues in Linear Regression

Overfitting vs Underfitting



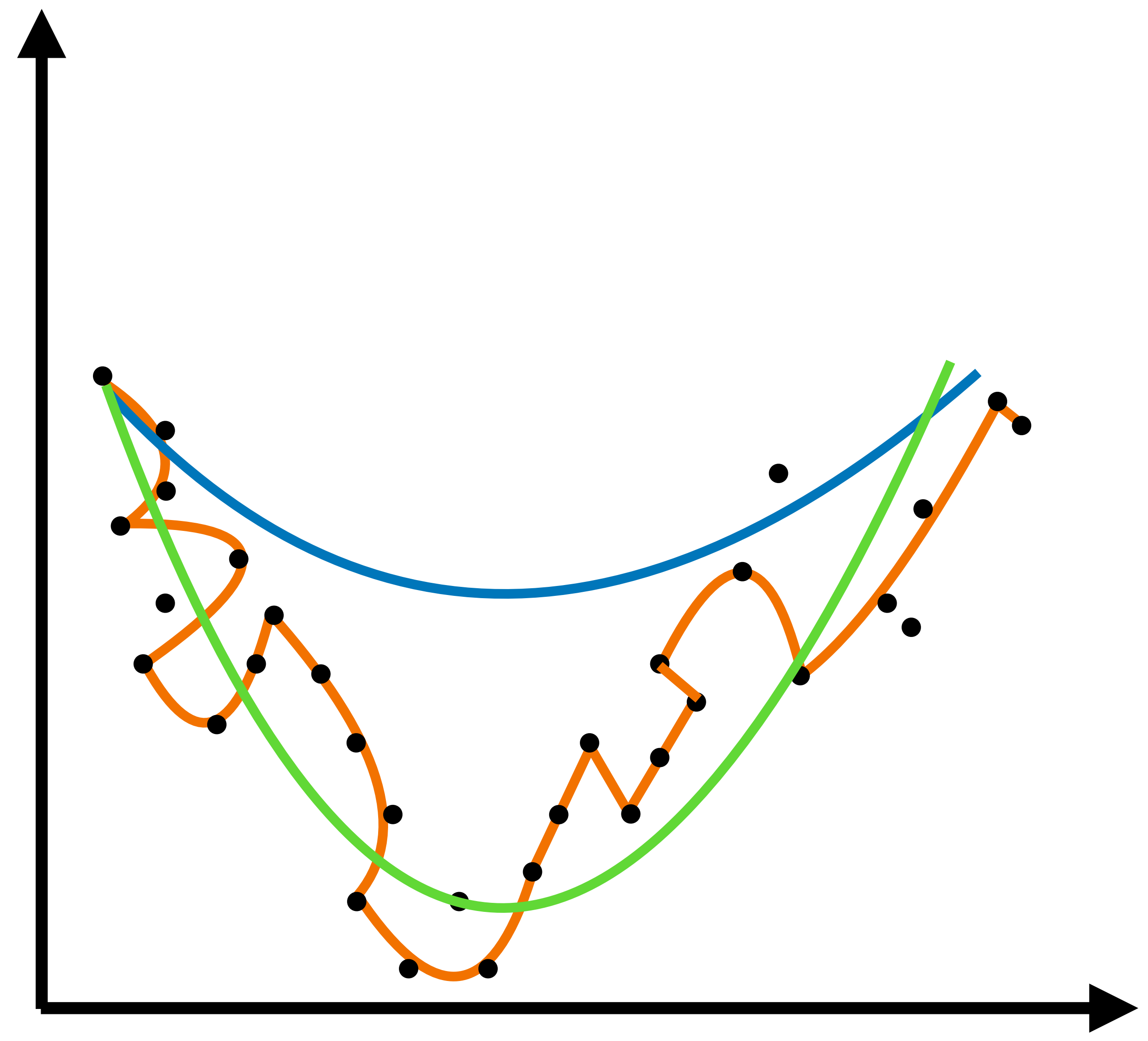
Practical Issues in Linear Regression

Overfitting vs Underfitting

The blue model is **underfitting** the data

The orange model is **overfitting** the data

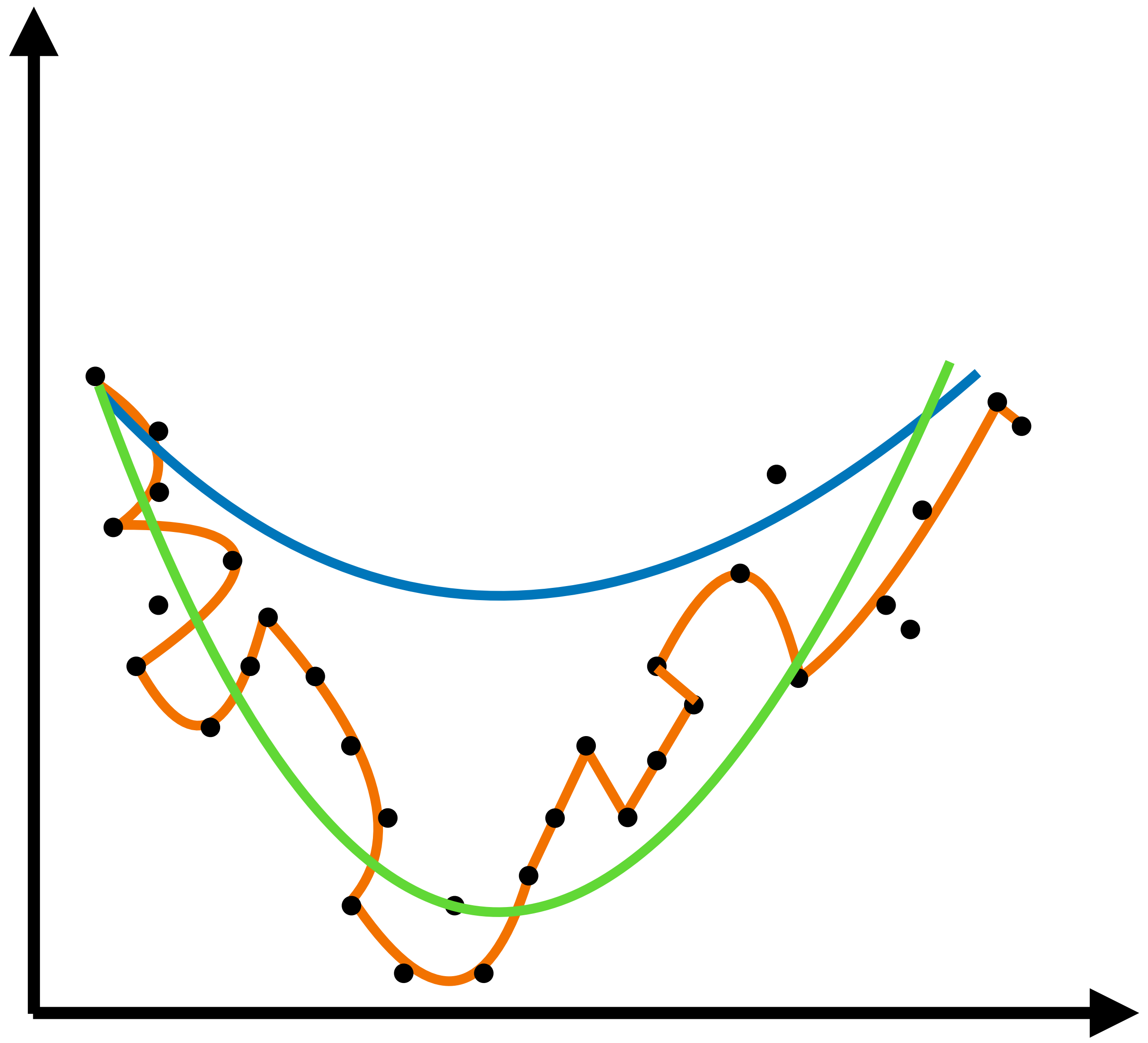
The green model is a good fit of the data



Practical Issues in Linear Regression

Underfitting

- What is happening?
    - The model is too simple to be able to capture the data
- How do you identify it?
    - Training loss is **high**
    - Test loss is **high**
- Solutions
    - Add more features
    - Add polynomial features $(x_1^2, x_2^2, x_1x_2, \cdots)$
    - Use a more complex model

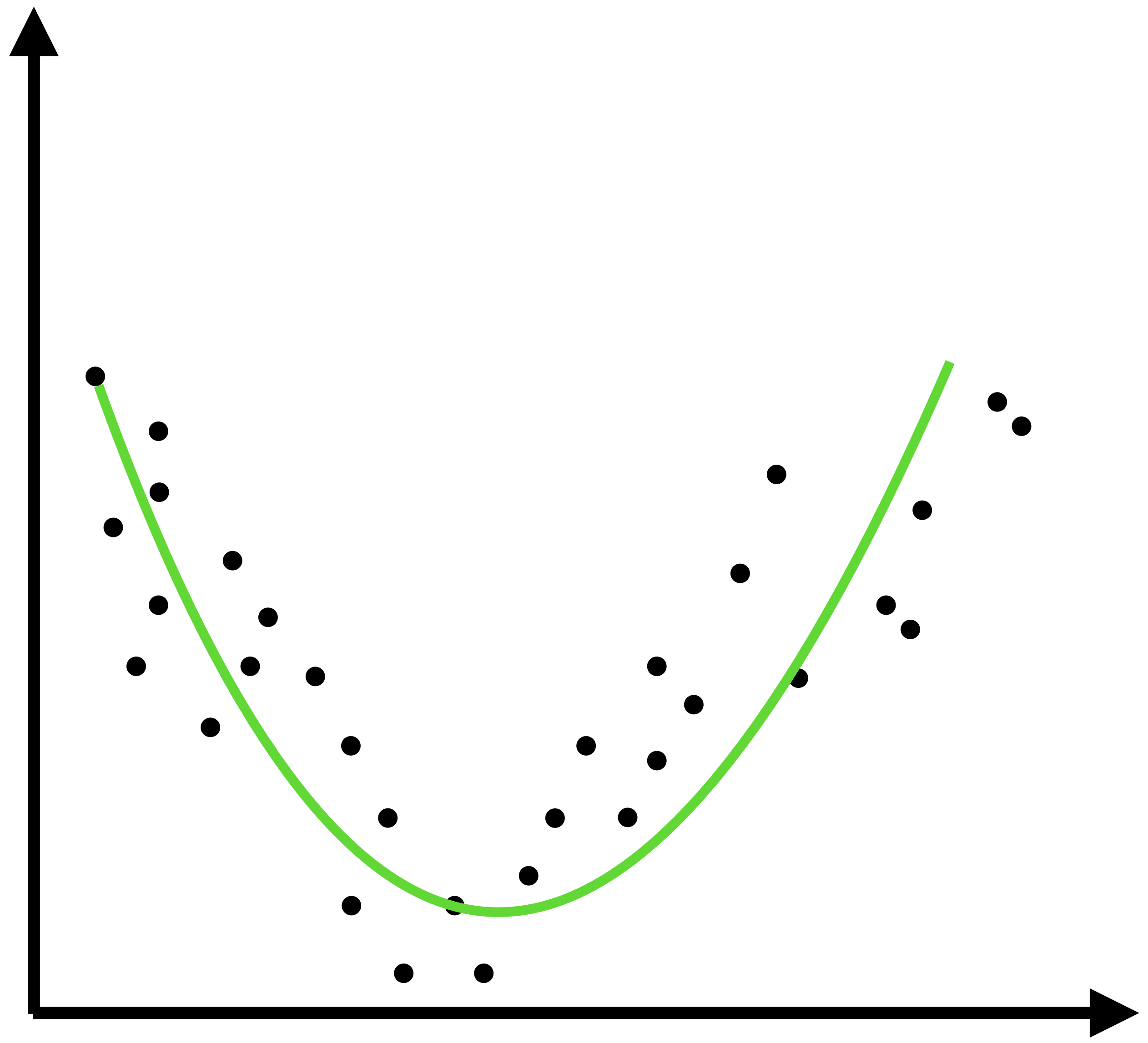


Practical Issues in Linear Regression

Quick Aside

- Add polynomial features $(x_1^2, x_2^2, x_1x_2, \cdots)$

$$f_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_1^2$$



Practical Issues in Linear Regression

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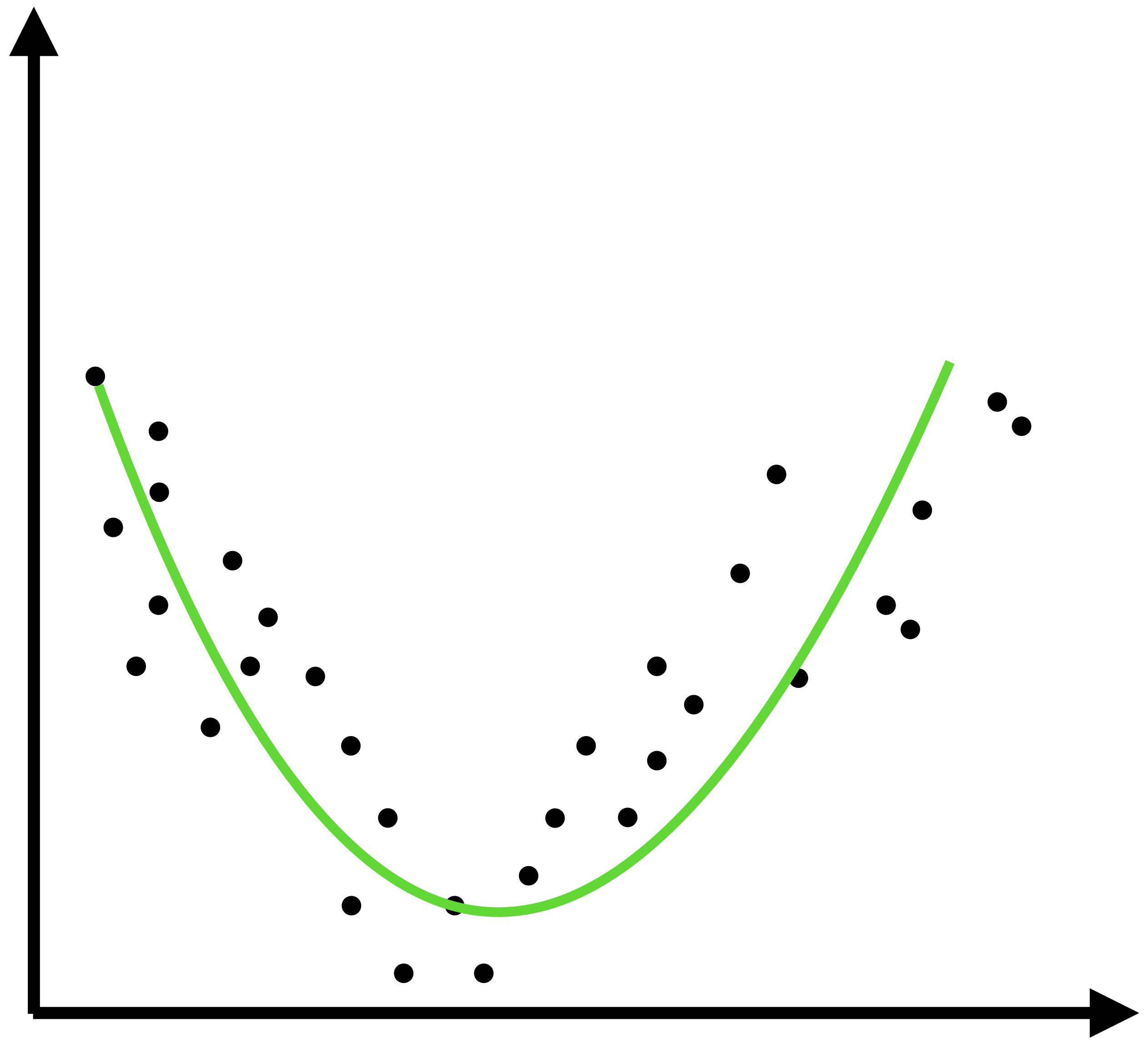


What about these models?

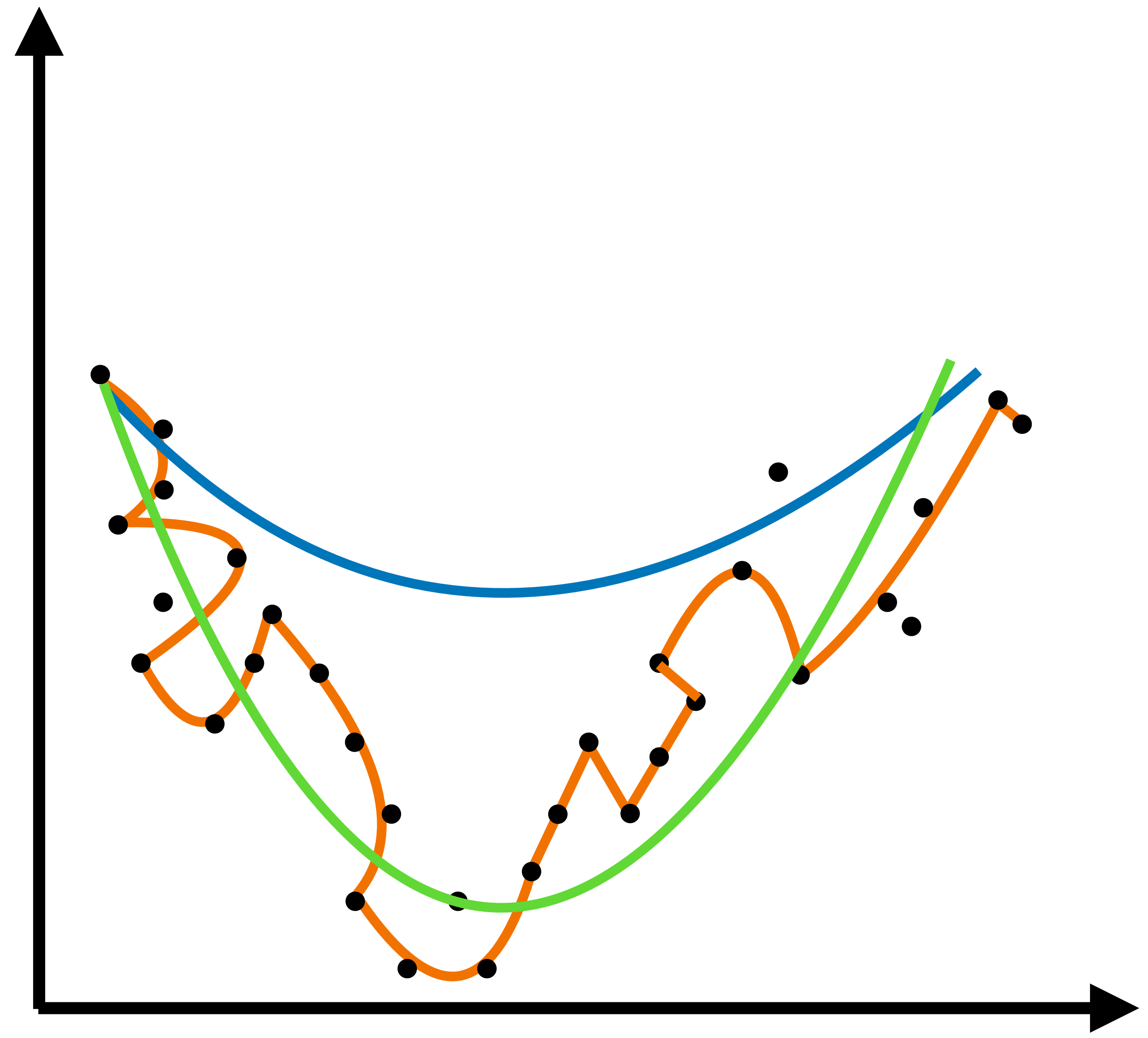
$$f_\theta(x) = \theta_0^{x_0} + \theta_1^{x_1}$$

$$f_\theta(x) = x_0^{\theta_0} + x_1^{\theta_1}$$

Practical Issues in Linear Regression

Overfitting

- What is happening?
    - The model is too complex, so it learns the noise distribution and outliers and hence does not generalize well to new data points
- How do you identify it?
    - Training loss is **low**
    - Test loss is **high**
    - Coefficients have **large** magnitudes
- Solutions
    - Regularization (L_1, L_2)
    - Cross-validation for model selection
    - Reduce number of features
    - Get more training data



Practical Issues in Linear Regression

A more mathematical look - Bias / Variance Tradeoff

Every model's prediction error/loss can be decomposed into three parts:

$$\text{Expected Loss} = \text{Bias}^2 + \text{Variance} + \text{Irreducible Noise}$$

Practical Issues in Linear Regression

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Error from wrong assumptions due to the model being too simple

Practical Issues in Linear Regression

A more mathematical look - Bias / Variance Tradeoff

Every model's prediction error/loss can be decomposed into three parts:

Expected Loss = Bias² + Variance + Irreducible Noise

Error from high sensitivity to each data point and noise due to the model being too complex

Practical Issues in Linear Regression

A more mathematical look - Bias / Variance Tradeoff

Every model's prediction error/loss can be decomposed into three parts:

$$\text{Expected Loss} = \text{Bias}^2 + \text{Variance} + \text{Irreducible Noise}$$

Inherent randomness in data. Cannot be removed.

Practical Issues in Linear Regression

Bias / Variance Tradeoff

Why is it called a **tradeoff**?

Practical Issues in Linear Regression

Bias / Variance Tradeoff

Why is it called a **tradeoff**?

Model Complexity	Bias	Variance	Train Error	Test Error
Too Simple	High	Low	High	High

Practical Issues in Linear Regression

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Practical Issues in Linear Regression

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Practical Issues in Linear Regression

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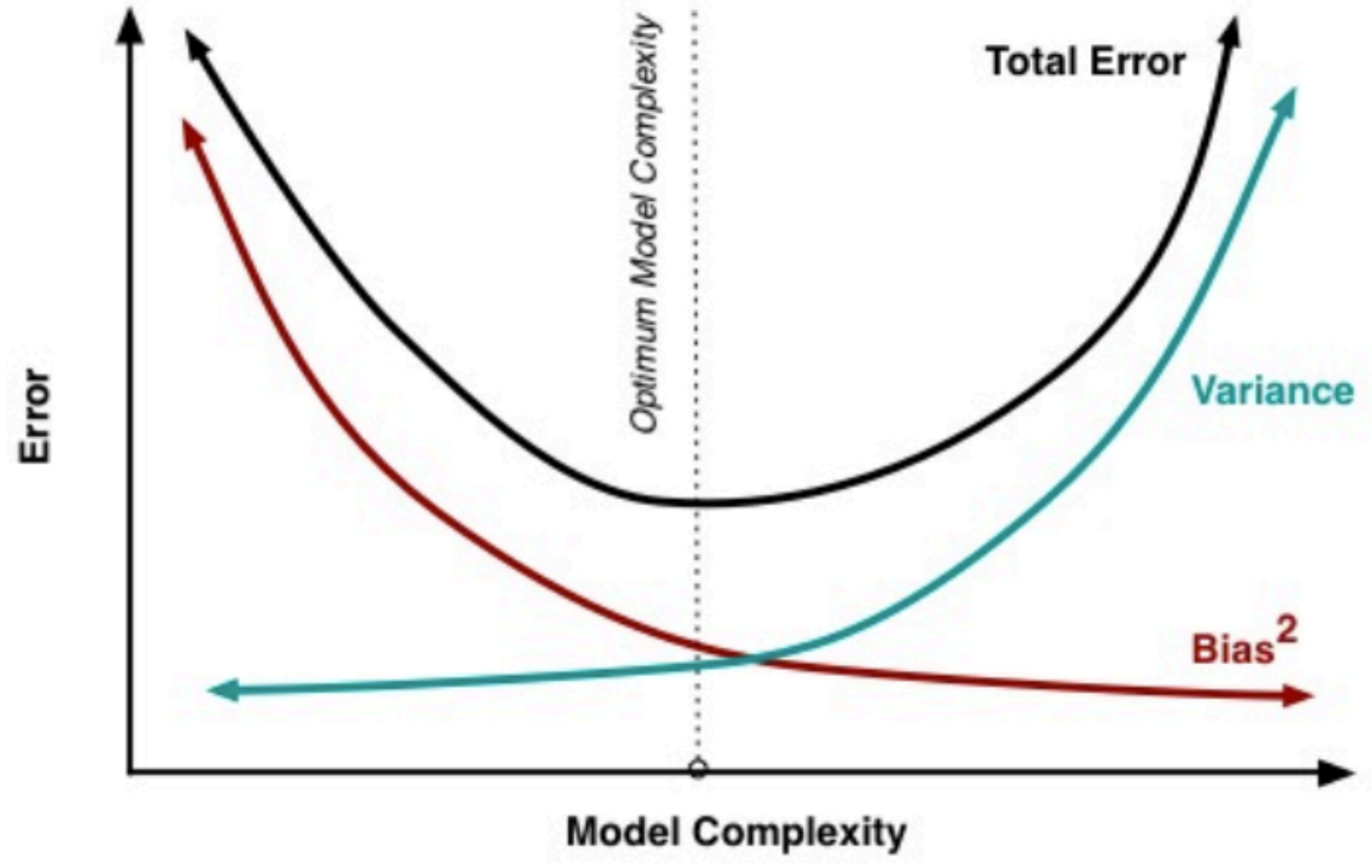
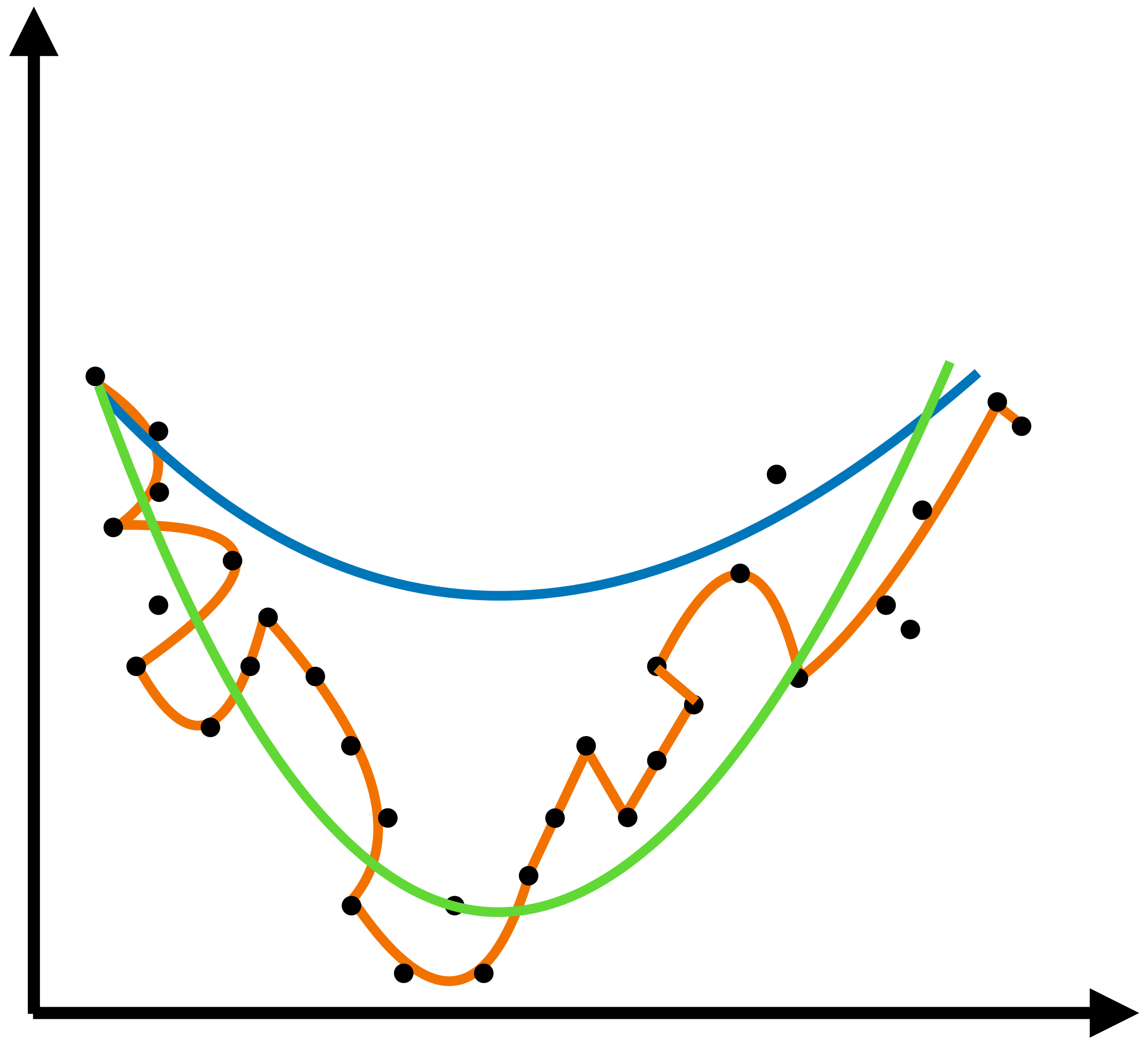


Practical Issues in Linear Regression

Regularization

- Regularization explicitly trades bias for variance.

$$L(\theta) = \frac{1}{m} \sum (Y - X\theta)^2$$



Practical Issues in Linear Regression

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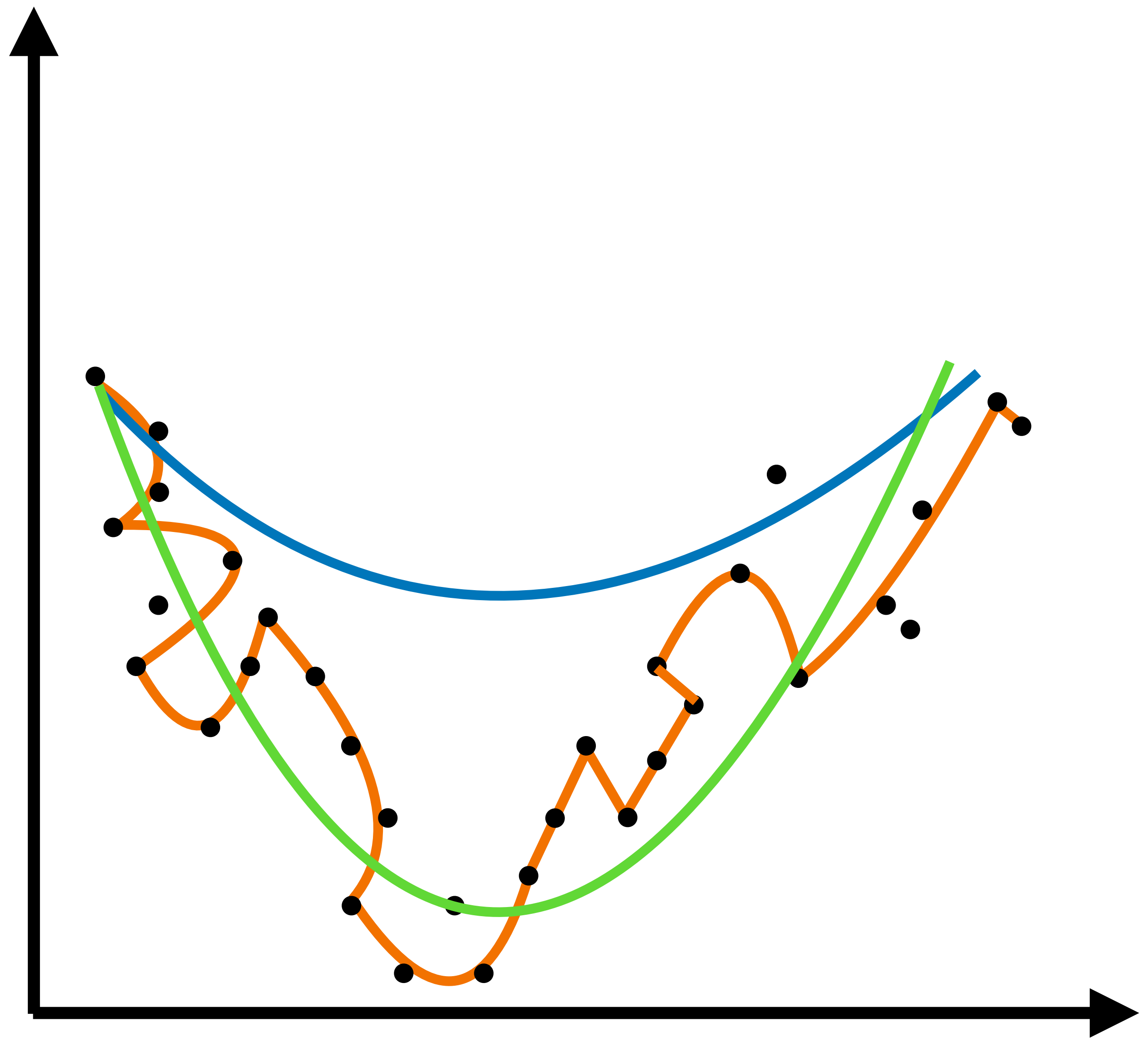
$$L(\theta) = \frac{1}{m}\sum (Y - X\theta)^2 + \lambda||\theta||^2$$



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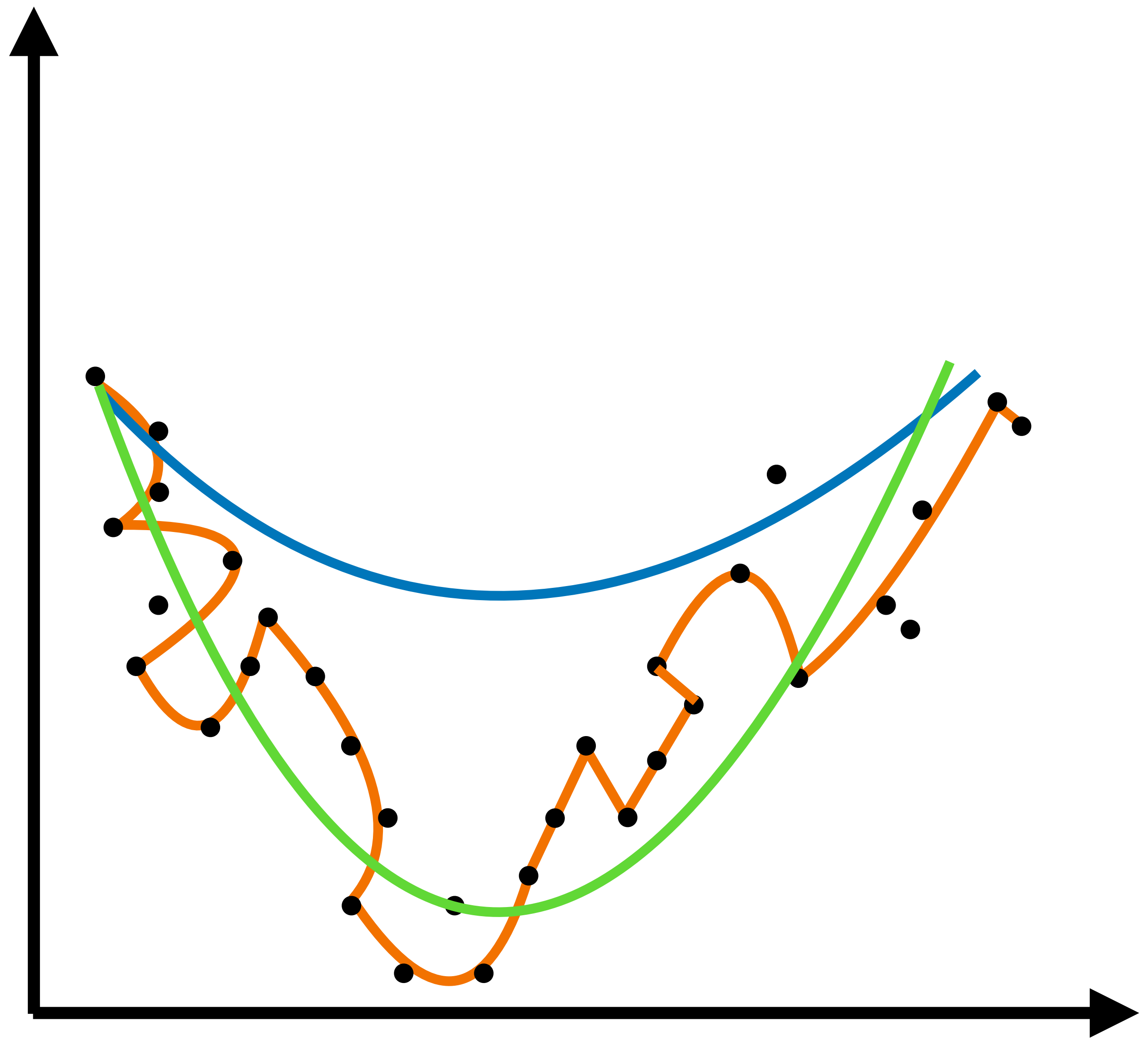
$$\theta = (X^T X + \lambda I)^{-1} X^T Y$$



Practical Issues in Linear Regression

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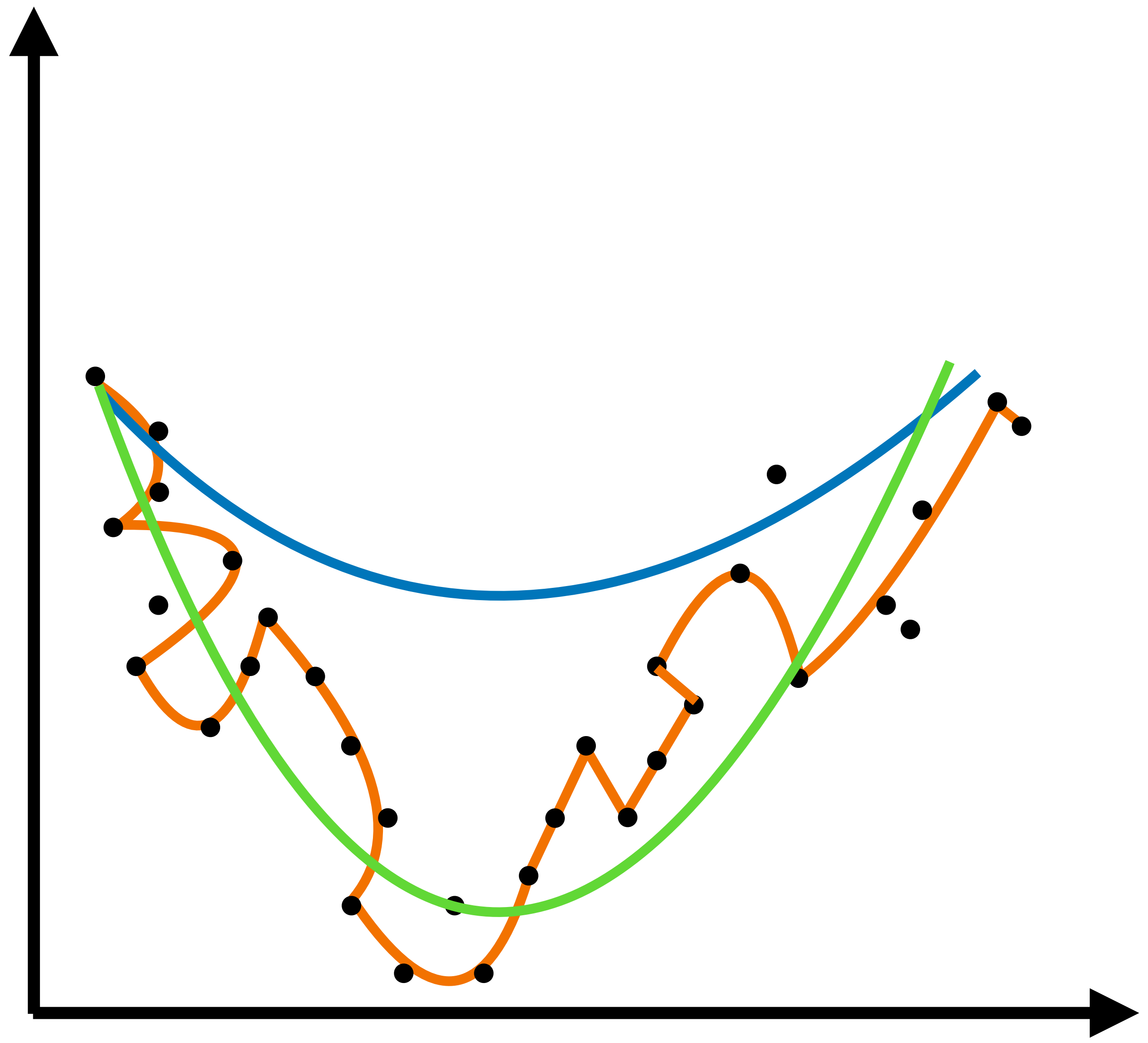

- As λ increases:
  - Coefficients shrink toward zero
  - Bias increases (we're constraining the model)
  - Variance decreases (less sensitive to data)
  - At some $\lambda *$, test error is minimized

Practical Issues in Linear Regression

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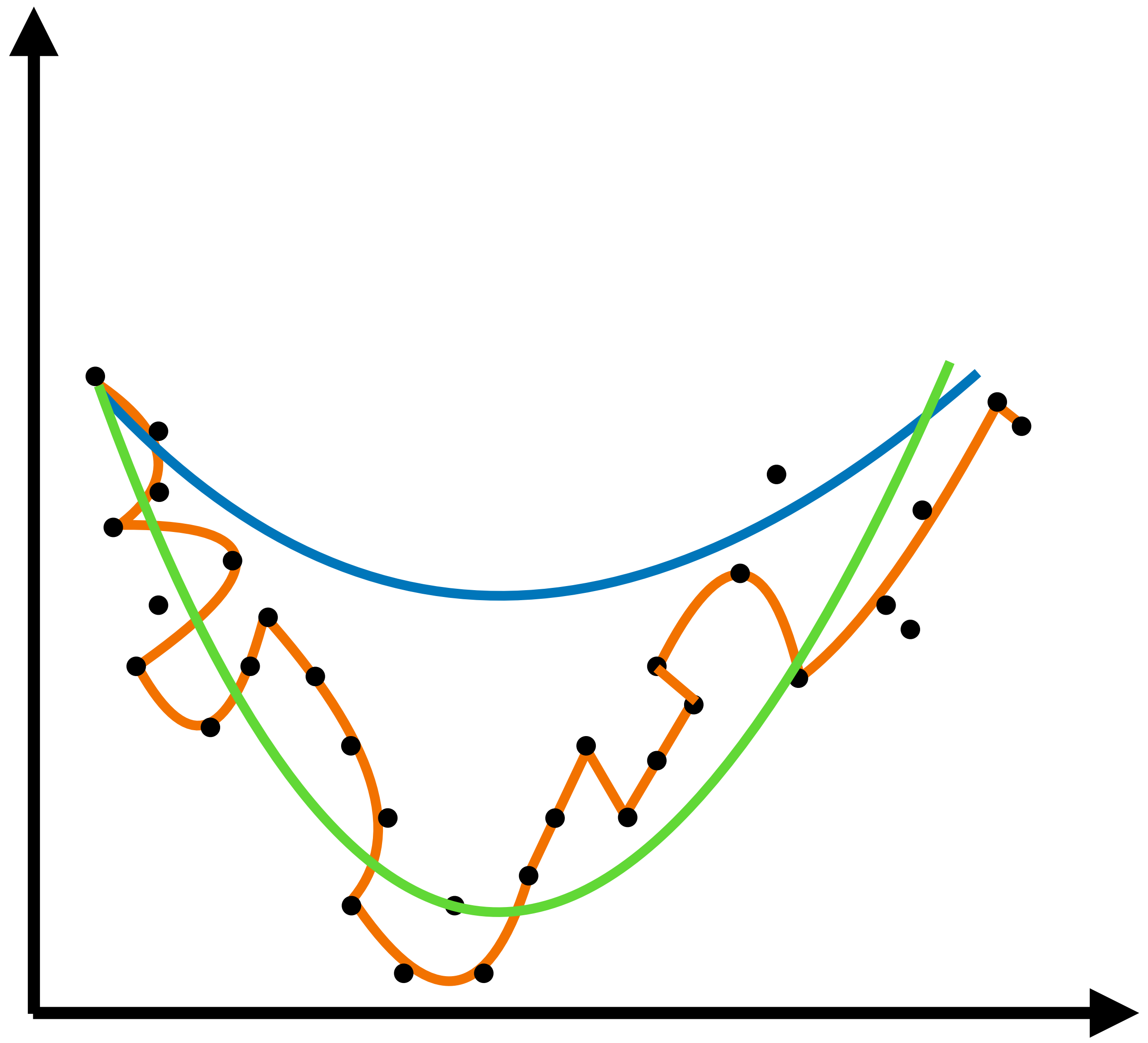
These sort of parameters are usually called **hyper-parameters**

They are **not learnable** but are human defined



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Feature Normalization

Why Normalize?

- If feature x_1 ranges from 0 to 1 and feature x_2 ranges from 0 to 1,000,000, this could lead to numerical instability in the solving process
  - This is particular relevant to gradient descent
- Regularization unfairness
  - If x_2 is much larger, θ_2 must be much smaller to produce similar predictions.
  - The regularization penalty then affects features unequally based on arbitrary scale choices.
- Distance-based algorithms

Feature Normalization

Normalization Methods

1. Min-Max Normalization
2. Mean-Variance Normalization
3. Max-Absolute Normalization
4. Robust Normalization

Feature Normalization

Min-Max Normalization

- For every column in the input data, i.e., for each x_0, x_1, x_2, x_4 etc., this normalization method will scale each column to 0 and 1

$$x' = \frac{x - min(x)}{max(x) - min(x)}$$

- This method preserves zero entries in sparse data

- But is very sensitive to **outliers**

Feature Normalization

Mean-Variance Normalization

- For every column in the input data, i.e., for each x_0, x_1, x_2, x_4 etc., this normalization method will scale to have mean 0 and standard deviation 1

$$x' = \frac{x - \mu(x)}{\sigma(x)}$$

- Most common in practice
- Less sensitive to outliers than min-max
- Does not bound the range to 0 and 1

Feature Normalization

Max-Absolute Normalization

- For every column in the input data, i.e., for each x_0, x_1, x_2, x_4 etc., this normalization method will scale each column to -1 and 1

$$x' = \frac{x}{|max(x)|}$$

- Good for sparse data since it preserves sparsity (zeros stay zero)

Feature Normalization

Robust Normalization

- For every column in the input data, i.e., for each x_0, x_1, x_2, x_4 etc., this normalization method will scale each column as

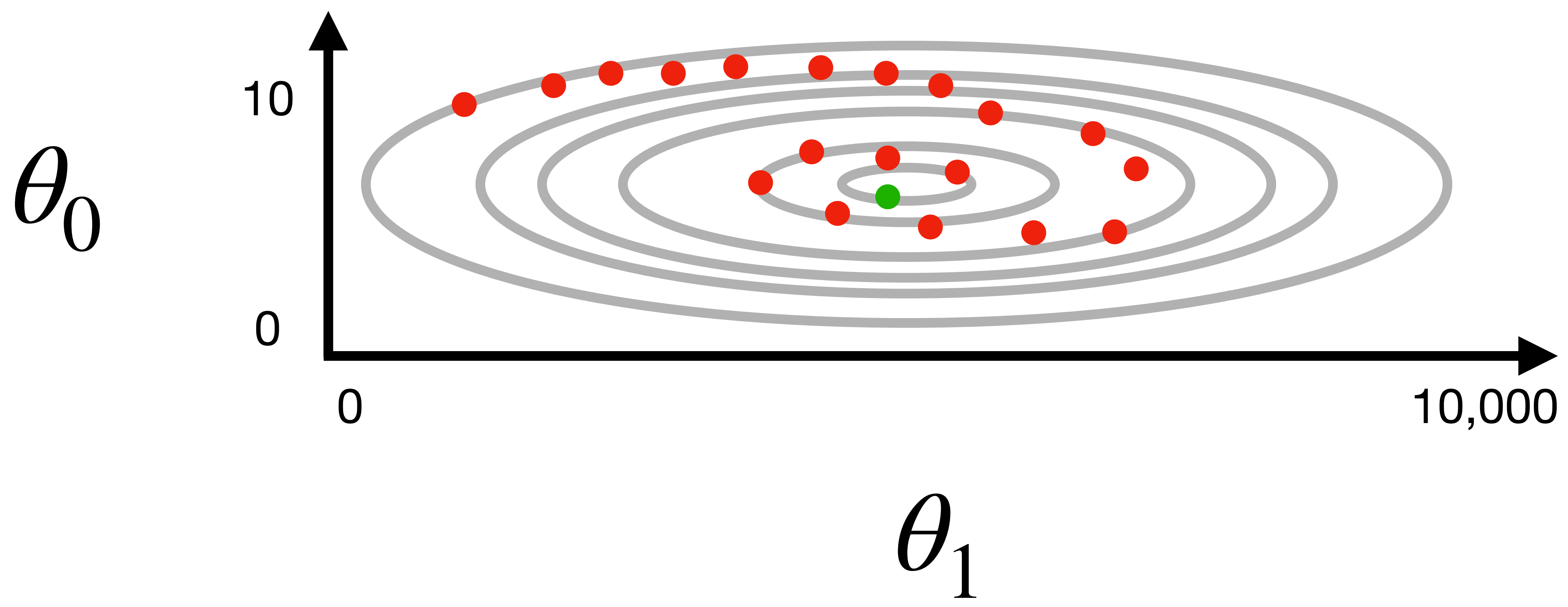
$$x' = \frac{x - median(x)}{IQR(x)}$$

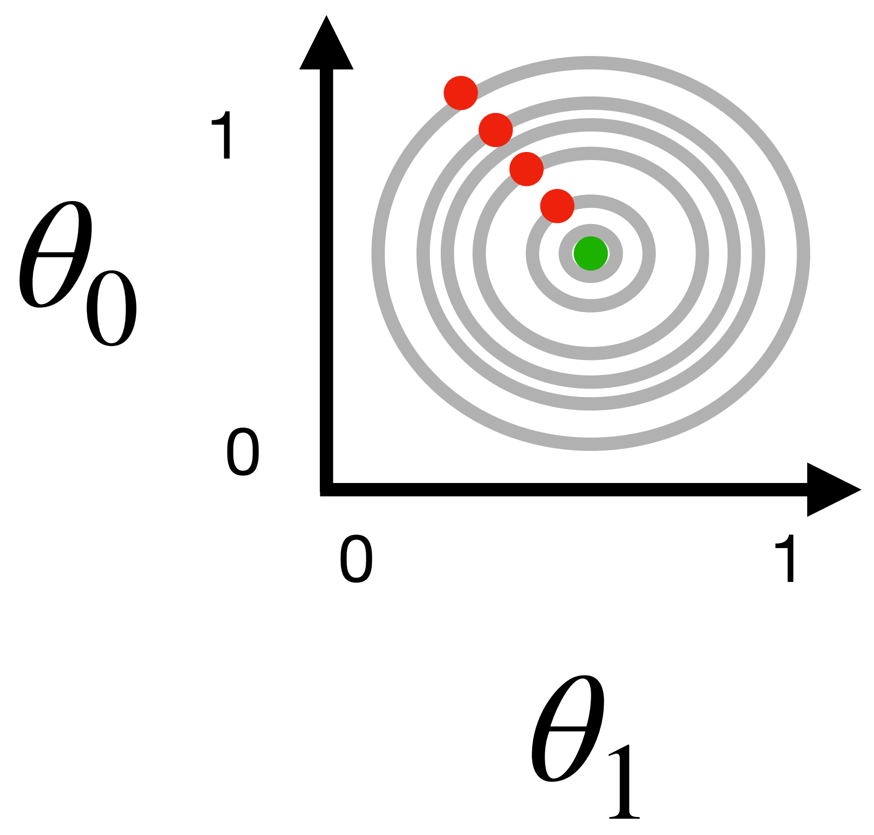
- Robust to outliers
- Use when data has many outliers

Optimizing Loss Functions

Gradient Descent - Practical Fixes

- Feature Scaling
  - Remember we want all input features $x_1, x_2 \cdots x_n$ to be in similar ranges
  - When features have different scales, the loss surface becomes elongated (ill-conditioned).





This dramatically accelerates the optimization process

This also allows having one single learning rate for all parameters

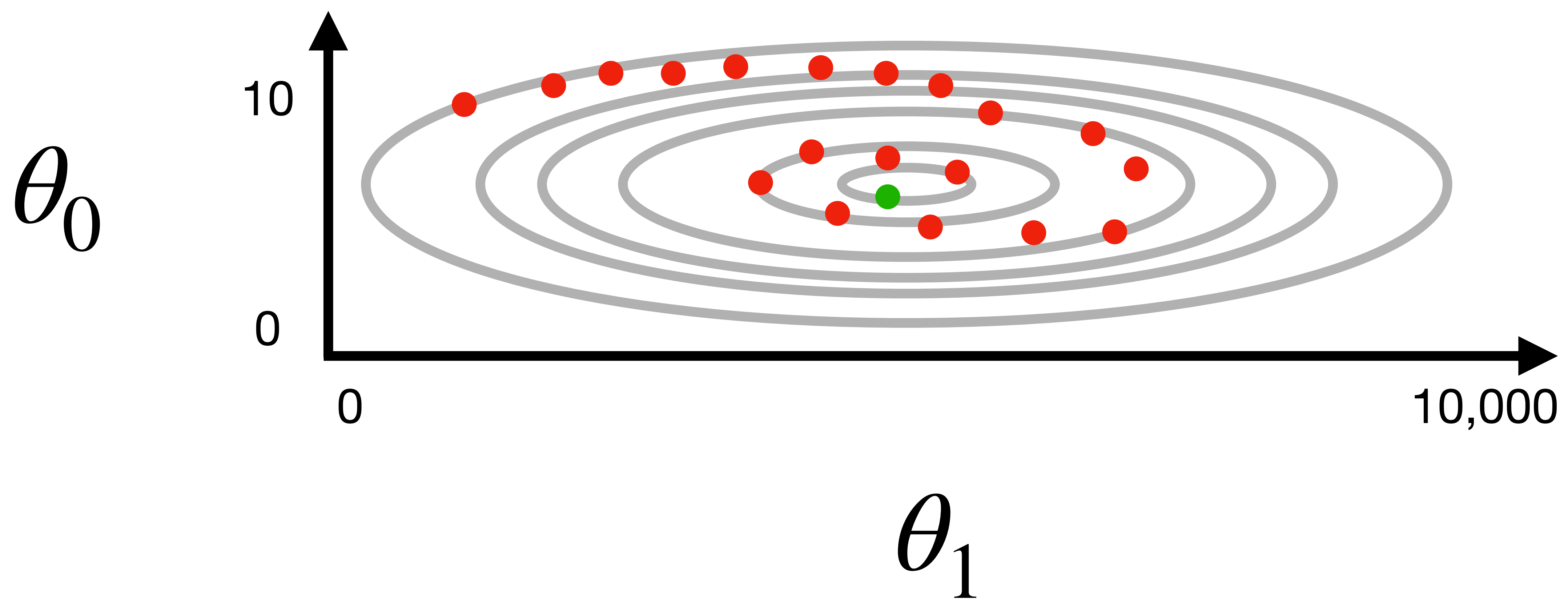
Optimizing Loss Functions

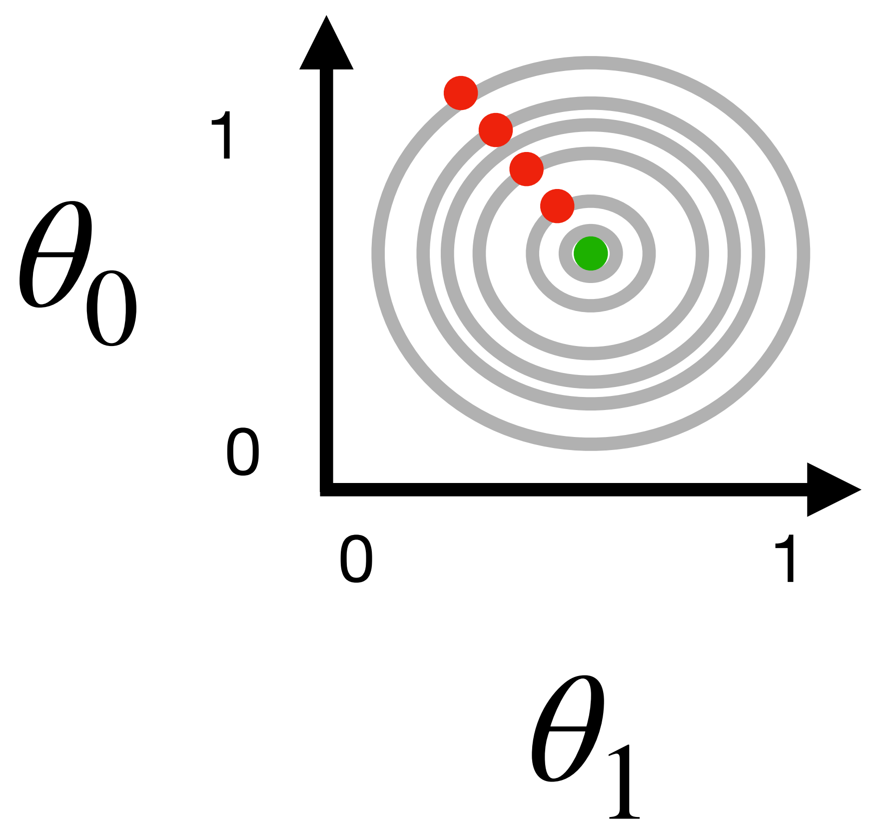
Gradient Descent - Practical Fixes

- Feature Scaling

NOTE: Scaling parameters (mean, standard deviation, min, max) must be computed **only on training data** and then applied to validation and test data to prevent data leakage.

  - Remember we want all input features $x_1, x_2 \cdots x_n$ to be in similar ranges
  - When features have different scales, the loss surface becomes elongated (ill-conditioned).





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Questions